Econ 626: Quantitative Methods II **Fall 2018**

Lecture 8: Review Session #8

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

Date: Date: September 7, 2018

8.1 Linear DE

$$\varphi_n(L) = 1 + a_{1,n}L + ... + a_{k,n}L^k \tag{8.1}$$

$$\begin{cases} \varphi_n(L)x_n = g_n, & \text{[C]} \\ \varphi_n(L)x_n = 0, & \text{[H]} \end{cases} \iff \begin{cases} x_n + a_{1,n}x_{n-1} + ... + a_{k,n}x_{n-k} = g_n, & \text{[C]} \\ x_n + a_{1,n}x_{n-1} + ... + a_{k,n}x_{n-k} = 0, & \text{[H]} \end{cases}$$

Theorem 8.1 *Consider a k-th order linear DE*

(a) $\{x_n\}$ *is a general solution to [C]* $\iff$ $x_n = x_{h,n} + x_{p,n}$, *where* $\begin{cases} \{x_{h,n}\} & \text{– general solution to [H]} \\ \{x_{p,n}\} & \text{– general solution to [C]} \end{cases}$

(b) $\{H\}$ *is a vector space.*

(c) $\dim\{H\} = k$.

(d) Let $\{x_n^1\}, ..., \{x_n^k\}$ *be solutions of [H] that satisfies* $\begin{cases} x_0^1 = 1, \\ \\ \end{cases}$

Then $\{\{x_n^1\}, ..., \{x_n^k\}\}$ *is a basis of* $\{H\}$.

Proof: Follows from the arguments similar to Theorem 3.2-3.4 + COrollary exercise. ■

8.2 Linear DE with constant coefficients

$$\varphi_n(L) = 1 + a_1L + ... + a_kL^k \tag{8.2}$$

$$\begin{cases} \varphi_n(L)x_n = g_n, & \text{[C]} \\ \varphi_n(L)x_n = 0, & \text{[H]} \end{cases} \iff \begin{cases} x_n + a_1x_{n-1} + ... + a_kx_{n-k} = g_n, & \text{[C]} \\ x_n + a_1x_{n-1} + ... + a_kx_{n-k} = 0, & \text{[H]} \end{cases}$$

Definition 8.2 $\lambda^k + a_1\lambda^{k-1} + ... + a_{k-1}\lambda + a_k = 0$ *[CE] is the* <u>*characteristic equation*</u> *corresponding to [H] (Assume* $a^k \neq 0$*).*

[1]Visit `http://www.luzk.net/misc` for updates.

Theorem 8.3 *λ is a solution to [CE] $\iff$ $\{\lambda^n\}, \lambda \neq 0$, is a solution to [H].*

Proof: "$\Longrightarrow$"

$\lambda^k + a_1\lambda^{k-1} + ... + a_{k-1}\lambda + a_k = 0$ multiplied by $\lambda^{n-k} \Longrightarrow$

$$\lambda^n + a_1\lambda^{n-1} + ... + a_{k-1}\lambda^{n-k+1} + a_k\lambda^{n-k} = 0$$

"$\Longleftarrow$"

$\lambda^n + a_1\lambda^{n-1} + ... + a_{k-1}\lambda^{n-k+1} + a_k\lambda^{n-k} = 0 \Longrightarrow \lambda^{n-k}(...) = 0 \Longrightarrow$True. ∎

Forming the basis of $\{H\}$

Suppose $\lambda_1, ..., \lambda_n$ are roots of [CE]

1. $\lambda_j \in \mathbb{R}$, distinct from other roots, then take $\{\lambda_j^n\}$
2. $\underbrace{\lambda_j, ..., \lambda_{j+m-1}}_{m \text{ terms}} \in \mathbb{R}$, equal real roots, then take $\{\lambda_j^n\}$, $\{n\lambda_j^n\}$, ..., $\{n^{m-1}\lambda_j^n\}$.
3. $$\begin{cases} \lambda_j = a_j + b_j i \Longrightarrow (a_j + b_j i)^n = (\underbrace{|z|}_{\Gamma_j} e^{i\theta_j})^n = \Gamma_j e^{in\theta_j} = \Gamma_j^n(\cos(n\theta_j) + i\sin(n\theta_j)) \\ \lambda_{j+1} = a_j - b_j i \Longrightarrow \qquad \Rightarrow \Gamma_j^n(\cos(n\theta_j) - i\sin(n\theta_j)) \end{cases}$$
   Then $c_1\lambda_j^n + c_2\lambda_{j+1}^n \Longrightarrow \Gamma_j^n(c_j\cos(n\theta_j) + c_{j+1}\sin(n\theta_j))$
4. $$\begin{cases} \lambda_j = a_j + ib_j \\ \lambda_{j+1} = a_j - ib_j \\ \lambda_{j+2} = a_j + ib_j \\ \lambda_{j+3} = a_j - ib_j \end{cases} \Longrightarrow \Gamma_j^n(c_j\cos(n\theta_j) + c_{j+1}\sin(n\theta_j) + c_{j+2}n\cos(n\theta_j) + c_{j+3}n\sin(n\theta_j))$$

General solution to [H]

$X_{h,n}$ is a linear combination of basis functions. For example,

$$\begin{cases} x_n + a_1x_{n-1} + ... + a_8x_{n-8} + a_9x_{n-9} = 0, & \text{[C]} \\ \lambda^9 + a_1\lambda_8 + ... + a_8 + \lambda a_9 = 0, & \text{[H]} \end{cases}$$

$$\begin{cases} \lambda_1, ..., \lambda_9 - \text{roots of [CE]} \\ \lambda_1 \neq \lambda_2 \neq \lambda_3 \in \mathbb{R} \\ \lambda_3 = \lambda_4 = \lambda_5 \in \mathbb{R} \\ \lambda_6 = a + bi \\ \lambda_7 = a - bi \\ \lambda_8 = a + bi \\ \lambda_9 = a - bi \end{cases}$$

$\Longrightarrow$

$$x_{h,n} = c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n + c_4n\lambda_4^n + c_5n^2\lambda_5^n + \Gamma^n(c_6\cos(b\theta) + c_7\sin(b\theta)) + n\Gamma^n(c_8\cos(b\theta) + c_9\sin(b\theta))$$

where $\Gamma = \sqrt{a^2 + b^2}$.

Particular solution to [C]

$$\phi(L)x_n = g_n \quad \text{[C]}$$

g_n	Guess for $x_{p,n}$
C – constant	D – constant
b^n	Db^n
$\sin(At)$	$D\sin(At) + E\cos(At)$
$\cos(At)$	$D\sin(At) + E\cos(At)$
n^d	$c_0 + c_1 t + ... + c_d t^d$
sum or product of the above	sum or product of the above

Remark: If $x_{p,n}$ solves [H], then multiplt it by n. ■

Example:

$$\begin{cases} x_n + 2x_{n-1} + 2x_{n-2} = n^2, & \text{[C]} \\ \lambda^2 + 2\lambda + 2 = 0, & \text{[CE]} \end{cases}$$

$$\Longrightarrow \begin{cases} \lambda_1 = -1 + i \\ \lambda_2 = -1 + i, \end{cases} \qquad \begin{cases} a = -1 \\ b = 1 \end{cases} \quad \Longrightarrow \Gamma = \sqrt{2},\ \theta = atan2(1,-1) = \frac{3\pi}{4}$$

$$\Longrightarrow x_{h,n} = (\sqrt{2})^n \left[c_1 \cos(\tfrac{3\pi n}{4}) + c_2 \sin(\tfrac{3\pi n}{4})\right]$$

$$\text{Guess for } x_{p,n} = b_0 + b_1 n + b_2 n^2 \Longrightarrow \begin{cases} b_0 = ... \\ b_1 = ... \\ b_2 = ... \end{cases}$$

... ■

Theorem 8.4 *(a) If $1 + a_1 + ... + a_k \neq 0$, 0 is a unique equilibrium point of [H]. Otherwise, any $z \in \mathbb{C}$ is an unstable equilibrium point.*

(b) 0 is table $\iff \Gamma_j < 1\ \forall j$, where $\Gamma_j = \sqrt{a_j^2 + b_j^2}$, $\lambda_j = a_j + ib_j$, $\lambda_1, ..., \lambda_k$ are roots of [CE].

Proof:

(a) $\tilde{x}$ is an equilibrium point $\iff x_n = \tilde{x}$ for all $n \Longrightarrow \tilde{x} + a_1\tilde{x} + ... + a_k\tilde{x} = 0 \Longrightarrow \tilde{x}(1 + a_1 + ... + a_k) = 0$

(b) $n^{m_j}\Gamma_j^n(\cos(n\theta_j) - i\sin(n\theta_j))$ (General form of a basis function for [H])

The conclusion follows from the argument in Theorem 6.3

■

Theorem 8.5 (Shur) *...*

8.3 Additional Topics

Phase diagram

$x_n = f(x_{n-2})$

`[insert a graph here]`

Remark: $f'(\tilde{x}) < 1 \Longrightarrow \tilde{x}$ is locally asymptotically stable.

$f'(\tilde{x}) > 1 \Longrightarrow \tilde{x}$ is unstable. ■

Linear Systems

$$\begin{cases} x_n = A_n x_{n-1} + B_n u_n & [C] \\ x_n = A_n x_{n-1} & [H] \end{cases}$$

Suppose $x_0 = x^0$, then

$$\begin{aligned} x_1 &= A_1 x_0 + B_1 u_1 \\ x_2 &= A_2 A_1 x_0 + A_2 B_1 u_1 + B_2 u_2 \\ &\vdots \\ x_n &= \left(\prod_{s=0}^{n-1} A_{n-s}\right) x_0 + \sum_{k=1}^{n} (\prod_{s=0}^{n-k-1} A_{n-s}) B_k u_k \end{aligned}$$

Constant coefficients

$$x_n = Ax_{n-1} + B_n u_n \Longrightarrow x_n = A^n x_0 + \sum_{k=1}^{n} A^{n-k} B_k u_k$$

Stability

$x_n = Ax_{n-1}$

$\tilde{x}$ is an equilibrium point $\iff$ $\tilde{x} = A\tilde{x}$.

$\Longrightarrow$The set of equilibrium points is $H_A(1) = \{x \in \mathbb{C}^n \mid x = Ax\}$. (Eigenspace w.r.t. 1)

If 1 is not an eigenvalue of A, then $\tilde{x} = 0$ is a unique equilibrium point.

Theorem 8.6 *$\tilde{x}$ is a stable equilibrium point of [H] $\iff$ moduli of all eigenvalues of A are less than 1.*

Proof: Note that $A = PDP^{-1}$ if diagonalizable. ■

Theorem 8.7 *If $A \in \mathbb{R}^{n\times m}$, $A = (a_{ij})$. Then $\sum_{j=1}^{n} |a_{ij}| < 1 \forall i \Longrightarrow$all eigenvalues with moduli < 1.*

Theorem 8.8 *$x_n = f(x_{n-1})$, $f : \mathbb{C}^n \to \mathbb{C}^n$. Suppose $\tilde{x}$ is an equilibrium point, i.e. $\tilde{x} = f(\tilde{x})$. if $Df(\tilde{x})$ has all eigenvalues with moduli ¡1, then $\tilde{x}$ is locally asymptotically stable.*

Proof: $x_n = f(x_{n-1}) \approx \underbrace{f(\tilde{x})}_{=\tilde{x}} + Df(\tilde{x})(x_{n-1} - \tilde{x}) \Longrightarrow \underbrace{x_n - \tilde{x}}_{y_n} \approx Df(\tilde{x}) \underbrace{(x_{n-1} - \tilde{x})}_{y_{n-1}}$. ■