Econ 626: Quantitative Methods II	**Fall 2018**

Lecture 7: Review Session #7

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

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7.1 Systems of Linear ODE with Constant coeffiicients

Theorem 7.1

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad [C] \\ \dot{x}(t) = A(t)x(t) \qquad [H] \end{cases} \tag{7.1}$$

Then e^{At} is a fundamental matrix of [H]. $e^{A(t-t_0)}$ is a state transtition matrix.

Proof: e^{At} is invertible, $(e^{At})^{-1} = e^{-At}$. And $\frac{\mathrm{d}}{\mathrm{d}t}(e^{At}) = Ae^{At}$. By Theorem 6.9, we know $e^{A(t-t_0)} = e^{At}(e^{At_0})^{-1}$ is a state transtition matrix. ■

Theorem 7.2 *General solution to [H] is $x(t) = e^{A(t-t_0)}x(t_0)$.*

General solution to [C] is $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}B(s)u(s)\mathrm{d}s$.

Exercise: Find the general solution of

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

7.2 Stability of Systems of ODE

Theorem 7.3 *Consider a linear system $\dot{x}(t) = Ax(t)$*

(a) 0 is a unique equilibrium point $\iff \det A \neq 0$

(b) 0 is a stable equilibrium point $\iff$ all eigenvalues of A have negative real parts.

Proof: (a) Suppose $Ax = 0$

"$\Longrightarrow$" part: $Ax \neq 0$ for any $x \neq 0 \Longrightarrow \det(A) \neq 0$. Contradiction!

"$\Longleftarrow$" part: $Ax = 0 \Longrightarrow x = 0$ is the unique equilibrium point.

(b) Suppose A is diagonalizable [if not, a similar argument can be developed using Jordan decomposition]. So $A = PDP^{-1} \Longrightarrow \dot{x} = PDP^{-1}x \Longrightarrow \underbrace{P^{-1}\dot{x}}_{y} = D\underbrace{P^{-1}x}_{\dot{y}} \iff \dot{y} = Dy$.

[1]Visit http://www.luzk.net/misc for updates.

Note that $\lim_{t\to\infty} x(t) = 0 \iff \lim_{t\to\infty} y(t) = 0$.

Now look at $\dot{y} = Dy$:

$$\dot{y} = \begin{bmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \iff \dot{y}_j = d_j y_j,\ j = 1, ..., n.$$

$y_j(t) = c_j e^{\alpha_t t} = c_j e^{(a_t + ib_j)t}$. Then $\lim_t y_j(t) = 0 \iff a_j < 0$. Hence the conclusion follows. ■

Theorem 7.4 *Consider a non-linear system $\dot{x} = f(x)$, $f : \mathbb{Q}^n \to \mathbb{R}^n$. Suppose f is C^1 and $\tilde{x}$ is an equilibrium point. Then*

(a) all eigenvalues of $Df(\tilde{x})$ havenegative real parts $\Longrightarrow \tilde{x}$ is locally asymptotically stable.

(b) at least one eigenvalue of $Df(\tilde{x})$ has positive real part $\Longrightarrow \tilde{x}$ is unstable.

Proof: (For a rigorous argument, see Hartman-Grobman theorem.)

By Taylor theorem,

$$\underbrace{f(x)}_{\dot{x}} = \underbrace{f(\tilde{x})}_{=0} + Df(\tilde{x})(x - \tilde{x}) + \Gamma(x - \tilde{x})$$

and $\lim_{x\to\tilde{x}} \frac{\Gamma(x-\tilde{x})}{||x-\tilde{x}||} = 0$.

$$\Longrightarrow \dot{x} = Df(\tilde{x})(x - \tilde{x}) + \Gamma(x - \tilde{x})$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}(x - \tilde{x}) = Df(\tilde{x}) \underbrace{(x - \tilde{x})}_{y} + \Gamma(x - \tilde{x})$$

$$\Longrightarrow \dot{y} = Df(\tilde{x})y + \Gamma(y)$$

Then when x is close to $\tilde{x}$, or y is close to 0, the stability of the original system is determined by the stability of its linearized version: $\dot{y} = Df(\tilde{x})y$. The conclusion follows from Theorem 7.3. ■

Phase Diagrams

$$\begin{cases} \dot{x_1} = f_1(x_1, x_2) \\ \dot{x_2} = f_2(x_1, x_2) \end{cases} \tag{7.2}$$

Example:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = x_1 \end{cases} \iff \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} \tag{7.3}$$

`[Insert a phase diagram here]`

■ **Remark:** Saddle path correspondences to an eigenvector with eigenvalue that has negative real part. ■

`[The rest part is missing]`