Econ 626: Quantitative Methods II **Fall 2018**

Lecture 6: Review Session #6

Lecturer: Aliaksandr Zaretski *Scribes: Zhikun Lu*

Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

Date: Date: September 5, 2018

Theorem 6.1 *Consider*

$$x^{(n)} + a_{n-1}x^{(n-1)} + ... + a_1 x' + a_0 = 0 \qquad [H] \tag{6.1}$$

0 is a stable equilibrium point of [H] $\Longrightarrow a_0, a_1, ..., a_{n-1} > 0$.

Theorem 6.2 (Routh–Hurwitz) *See the scanned page.*

6.1 System of linear ODE

$$\underbrace{\dot{x}(t)}_{n\times 1} = \underbrace{A(t)}_{n\times n}\underbrace{x(t)}_{n\times 1} + \underbrace{B(t)}_{n\times m}\underbrace{u(t)}_{m\times 1} \tag{6.2}$$

Remark: Consider

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + ... + a_1(t)y' + a_0(t)y = g(t)$$

Let

$$\begin{cases} x_1 & \equiv y \\ x_2 & \equiv y' \\ & \vdots \\ x_n & \equiv y^{(n-1)} \end{cases} \Longrightarrow \begin{cases} \dot{x}_1 & \equiv x_2 \\ \dot{x}_2 & \equiv x_3 \\ & \vdots \\ \dot{x}_{n-1} & \equiv x_n \end{cases}$$

$$\Longrightarrow \dot{x}_n + a_{n-1}(t)x_n + ... + a_1(t)x_2 + a_0(t)x_1 = g(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \qquad B(t) = \begin{bmatrix} b_{11}(t) & \cdots & b_{1m}(t) \\ \vdots & & \vdots \\ b_{n1}(t) & \cdots & b_{nm}(t) \end{bmatrix}$$

[1] Visit http://www.luzk.net/misc for updates.

Then we have

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-2}(t) & -a_{n-1}(t) \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B(t)} \underbrace{g(t)}_{u(t)} \tag{6.3}$$

■

Theorem 6.3 *If all elements of $A(t), B(t), u(t)$ are continuous, then $\forall$ initial condition $x(t_0) = x^0$, $\exists$ a unique solution to [C]*

Remark:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad [C] \tag{6.4}$$

$$\dot{x}(t) = A(t)x(t) \qquad [H] \tag{6.5}$$

■

Theorem 6.4 *$\{H\}$ is a vector space.*

Proof: Excercise. ■

Theorem 6.5 $\dim\{H\} = n$.

Definition 6.6 *Let $\{x^1(t), ..., x^n(t)\}$ be a basis of $\{H\}$. Then*

$$\Phi(t) = (x^1(t), ..., x^n(t)) = \begin{bmatrix} x_1^1(t) & \cdots & x_1^n(t) \\ \vdots & & \vdots \\ x_n^1(t) & \cdots & x_n^n(t) \end{bmatrix} \tag{6.6}$$

where $\Phi(t)$ is a fundamental matrix of [H].

6.2 Systems of linear ODE

Remark:

1. There are infinitely many fundamental matrices of [H]
2. $\dot{\Phi}(t) \equiv (\dot{x}^1(t), ..., \dot{x}^n(t)) = \begin{bmatrix} \dot{x}_1^1(t) & \cdots & \dot{x}_1^n(t) \\ \vdots & & \vdots \\ \dot{x}_n^1(t) & \cdots & \dot{x}_n^n(t) \end{bmatrix}$

■

Lemma 6.7 *$\Phi(t)$ is a fundamental matrix of [H] $\iff$ $\dot{\Phi}(t) = A(t)\Phi(t)$ and $\Phi(t)$ is invertible for all t.*

Proof: "$\Longrightarrow$"

$$\dot{\Phi}(t) = (\dot{x}^1(t), ..., \dot{x}^n(t)) = (A(t)x^1(t), ..., A(t)x^n(t)) = A(t)(x^1(t), ..., x^n(t)) = A(t)\Phi(t) \tag{6.7}$$

"$\Longleftarrow$"

$\Phi(t)$ has linearly independent columns. Suppose $x(t)$ is a solution to [H] that satisfies $x(t_0) = x^0$. Consider $\tilde{x}(t) = \Phi(t)c$. Then $\tilde{x}(t_0) = \Phi(t_0)c \Longrightarrow c = \Phi(t_0)^{-1}\tilde{x}(t_0)$.

Suppose $\tilde{x}(t_0) = x^0$. Then $\tilde{x}(t) = \Phi(t)\Phi(t_0)^{-1}x^0$. Then

$$\dot{\tilde{x}}(t) = \dot{\Phi}(t)\Phi(t_0)^{-1}x^0 = A(t)\underbrace{\Phi(t)\Phi(t_0)^{-1}x^0}_{\tilde{x}(t)} = A(t)\tilde{x}(t) \tag{6.8}$$

Hence $\tilde{x}(t)$ is a solution to [H]. By uniqueness,

$$\tilde{x}(t) = x(t) = \Phi(t)\underbrace{\Phi(t_0)^{-1}x^0}_{c} \tag{6.9}$$

Hence, $\Phi(t)$ has a basis of $\{H\}$ as columns $\Longrightarrow$fundamental matrix. ∎

Definition 6.8 *$\Phi(t, t_0)$ is a <u>state transition matrix of</u> [H] if*

1. $\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$
2. $\Phi(t_0, t_0) = I_n$

Theorem 6.9 *If $\Phi(t)$ is a fundamental matrx of [H], then $\Phi(t, t_0) = \Phi(t)\Phi(t_0)^{-1}$ is a state transition matrx.*

Proof:

1. $\dot{\Phi}(t, t_0) = \dot{\Phi}(t)\Phi(t_0)^{-1} = A(t)\Phi(t)\Phi(t_0) = A(t)\Phi(t, t_0)$
2. $\Phi(t_0, t_0) = \Phi(t_0)\Phi(t_0)^{-1} = I_n$

∎

Remark: State transition matrx is unique. **Proof:** Exercise. ∎

Theorem 6.10 *Suppose assumptions of theorem ???(7.1) holds. Then $x(t) = \Phi(t, t_0)x^0$ is the unique solution to [H] that satisfies $x(t_0) = x^0$.*

Proof: $x(t) = \Phi(t)c$ for some $c \in \mathbb{R}^n$ $\Longrightarrow x(t_0) = \Phi(t_0)c \Longrightarrow c = \Phi(t_0)^{-1}x(t_0)$

$\Longrightarrow x(t) = \Phi(t)\Phi(t_0)^{-1}x(t_0) = \Phi(t, t_0)x(t_0)$ ∎

Remark: $x(t) = \Phi(t)c, \ c \in \mathbb{R}^n$ is the general solution to [H]. ∎

Theorem 6.11 *The general solution to [C] is*

$$x(t) = \Phi(t)c + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)\mathrm{d}s. \tag{6.10}$$

Hence, the solution that satisfies $x_{t_0} = x^0$ *is*

$$x(t) = \Phi(t,t_0)x^0 + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)\mathrm{d}s \tag{6.11}$$

6.3 Systems of linear ODE with constant coefficients

$$\dot{x}(t) = Ax(t) + B(t)u(t) \qquad [C] \tag{6.12}$$

$$\dot{x}(t) = Ax(t) \qquad [H] \tag{6.13}$$

Definition 6.12 *Let* $A \in \mathbb{R}^{n\times n}$*. Then* $e^A \equiv \sum_{k=0}^{\infty} \frac{A^k}{k!}$ *is the* <u>*matrix exponential*</u> *of* A.

Theorem 6.13 *Let* $A, B, C \in \mathbb{R}^{n\times n}$*, and* $c, d \in \mathbb{R}$.

1. *Suppose* $AB = BA$*. Then*
$$e^{cA}e^{dB} = e^{cA+dB}$$
In particular, $e^{cA}e^{dA} = e^{(c+d)A}$.
2. $\det(e^A) = e^{tr(A)}$*. Hence,* e^A *is invertible* $\forall A \in \mathbb{R}^{n\times n}$.
3. *Suppose* D *is diagonal, i.e.*
$$D = diag(d_i) = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$
Then
$$e^D = diag(e^{d_i}) = \begin{bmatrix} e^{d_1} & & 0 \\ & \ddots & \\ 0 & & e^{d_n} \end{bmatrix}.$$
In particular, $e^{0_{n\times n}} = I_n,\ e^{I_n} = eI_n$.
4. $(e^A)^{-1} = e^{-A}$.
5. *Let* $A = PJP^{-1}$*, where* J *– Jordan normal form, then*
$$e^A = Pe^JP^{-1}.$$
Suppose A *is diagonalizable,* $A = PDP^{-1}$*, then* $e^A = Pe^DP^{-1}$
6. $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{At}\right) = Ae^{At}$.

Proof:

- 1. follows from Binomial formula / Cauchy product.
- 2. follows from Jacobi's formula.
- 3. – 6. Exercise.
- For 4., use 1.

■