Econ 626: Quantitative Methods II	**Fall 2018**

Lecture 4: Review Session #4

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

4.1 Complex Numbers

$\mathbb{C}$ – the set of Complex numbers.

If $z \in \mathbb{C}$, then $Z = a + bi$, where a is the real part and b is the imaginary part.

Polar form: We use $|z| = \sqrt{a^2 + b^2}$ – modulus. Then $z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$.

If $\theta = \pi$, then $e^{i\pi} = -1$ (Euler's identity).

$$\Longrightarrow \begin{cases} a = |z|\cos\theta \\ b = |z|\sin\theta \end{cases} \Longrightarrow \frac{b}{a} = \tan\theta \tag{4.1}$$

Remark: To find θ based on a, b, use the $atan2$ function:

$$\theta = atan2(b, a) = \begin{cases} atan(\frac{b}{a}), & \text{if } a > 0 \\ atan(\frac{b}{a}) + \pi, & \text{if } a < 0 \wedge b \geq 0 \\ atan(\frac{b}{a}) - \pi, & \text{if } a < 0 \wedge b < 0 \\ \frac{\pi}{2}, & \text{if } a = 0 \wedge b > 0 \\ -\frac{\pi}{2}, & \text{if } a = 0 \wedge b < 0 \\ \text{undefined}, & \text{if } a = 0 \wedge b = 0 \end{cases} \tag{4.2}$$

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4.2 Solutions to the characteristic equation [CE]

Suppose $\lambda \in \mathbb{C}$ is a solution to [CE]. Then $\lambda = a + bi$. Then $e^{\lambda t}$ is a solution to [H].

$$e^{\lambda t} = e^{(a+bi)t} = e^{at}e^{ibt} = e^{at}(\cos(bt) + i\sin(bt)) \tag{4.3}$$

Also, $\bar{\lambda} = a - bi$ is a solution to [CE].

$$e^{\bar{\lambda}} = e^{(a-bi)t} = e^{at}(\cos(-bt) + i\sin(-bt)) = e^{at}(\cos(bt) - i\sin(bt)) \tag{4.4}$$

Consider $(e^{\lambda t}, e^{\bar{\lambda}t})$ – these are two independent solutions to [H]. Their linear combination

$$C_1e^{\lambda t} + C_2e^{\bar{\lambda}t} = e^{at}[(C_1 + C_2)\cos(bt) + i(C_1 - C_2)\sin(bt)] \tag{4.5}$$

$$= e^{at}[C_3\cos(bt) + C_4\sin(bt)], \quad C_3, C_4 \in \mathbb{C} \tag{4.6}$$

[1]Visit `http://www.luzk.net/misc` for updates.

4.3 Related solutions to [CE]

Suppose $\mu(\lambda) = m > 1$. Then the independent solutions associated to λ are $\{e^{\lambda t}, te^{\lambda t}, ..., t^{m-1}e^{\lambda t}\}$.

Intuition from 2nd order case

$$x'' + a_1 x' + a_0 x = 0 \qquad \text{[H]}$$

$$\lambda^2 + a_1\lambda + a_0 = 0 \qquad \text{[CE]}$$

$\mathcal{D} = a_1^2 - 4a_0$

$$\begin{cases} \mathcal{D} > 0 \implies \text{distinct real roots} \\ \mathcal{D} = 0 \implies \text{equal real roots} \\ \mathcal{D} < 0 \implies \text{complex conjugates} \end{cases} \tag{4.7}$$

Suppose $\mathcal{D} = 0$. Then $\lambda_{1,2} = \frac{-a_1}{2}$. We have solutions $e^{\lambda t}$, $te^{\lambda t}$.

Consider $x(t) = te^{\lambda t}$,

$$x'(t) = e^{\lambda t} + \lambda t e^{\lambda t} \tag{4.8}$$

$$x''(t) = 2\lambda e^{\lambda t} + \lambda^2 t e^{\lambda t} \tag{4.9}$$

Substitute into [H]

$$2\lambda e^{\lambda t} + \lambda^2 t e^{\lambda t} + a_1 e^{\lambda t} + a_1 \lambda t e^{\lambda t} + a_0 t e^{\lambda t} = 0 \tag{4.10}$$

$$2\lambda + \lambda^2 t + a_1 + a_1\lambda t + a_0 t = 0 \tag{4.11}$$

$$2(\frac{-a_1}{2}) + (\frac{-a_1}{2})^2 t + a_1 + a_1(\frac{-a_1}{2})t + a_0 t = 0 \tag{4.12}$$

$$(a_1^2 - 4a_0)t = 0 \tag{4.13}$$

General solution to [H]

Consider, for example, the following:

$$x^{(9)} + a_8 x^{(8)} + ... + a_1 x' + a_0 = 0 \qquad \text{[H]}$$

$$\lambda^9 + a_8\lambda^8 + ... + a_1\lambda + a_0 = 0 \qquad \text{[CE]}$$

Let $\lambda_1, ..., \lambda_9$ be the roots of [CE]. Suppose $\lambda_1, \lambda_2, ..., \lambda_5 \in \mathbb{R}$, $\lambda_6, ..., \lambda_9 \in \mathbb{C}$. Suppose

$$\begin{cases} \lambda_1 \neq \lambda_2 \neq \lambda_3 \\ \lambda_3 = \lambda_4 = \lambda_5 \\ \lambda_6 = \lambda_8 \\ \lambda_7 = \lambda_9 \end{cases} \tag{4.14}$$

Let $\lambda_6 = a + bi$. Then $\lambda_7 = a - bi, \lambda_8 = a + bi, \lambda_9 = a - bi$.

Then the general solution to [H] is

$$x_n(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t} + C_4 e^{\lambda_4 t} + C_5 e^{\lambda_5 t} + e^{at}(C_6\cos(bt) + C_7\sin(bt)) + te^{at}(C_8\cos(bt) + C_9\sin(bt)) \tag{4.15}$$

Particular solution to [C]

$$Lx(t) = g(t) \qquad \text{[C]} \tag{4.16}$$

Remark: If the guess for $x_p(t)$ solves [H], then multiply the guess by t. ■

$g(t)$	Guess for $x_p(t)$
C – constant	D – constant
$e^{\lambda t}$	$De^{\lambda t}$
t^Γ	$b_0 + b_1 t + ... + b_\Gamma t^\Gamma$
$\sin(bt)$	$D\sin(bt) + E\cos(bt)$
$\cos(bt)$	$D\sin(bt) + E\cos(bt)$
λ^t	$D\lambda^t$
sum or product of the above	sum or product of the above

Example:

$$\begin{cases} x'' + 2x' + 2x = t^2, & \text{[C]} \\ x'' + 2x' + 2x = 0, & \text{[H]} \\ \lambda^2 + 2\lambda + 2 = 0, & \text{[CE]} \end{cases} \tag{4.17}$$

$\mathcal{D} = 4 - 8 < 0$, $\lambda_{1,2} = -1 \pm i$. So $a = -1, b = 1$

$$x_h(t) = e^{-t}(C_1\cos(t) + C_2\sin(t)) \text{ – general solution to [H]} \tag{4.18}$$

Try

$$x_p(t) = b_0 + b_1 t + b_2 t^2 \Longrightarrow \begin{cases} x_p' = b_1 + 2b_2 t \\ x_p'' = 2b_2 \end{cases} \tag{4.19}$$

Substitute into [C]:

$$(2b_2) + 2(b_1 + 2b_2 t) + 2(b_0 + b_1 t + b_2 t^2) = t^2 \tag{4.20}$$

We need: $\begin{cases} 2b_2 = 1 & \Longrightarrow b_2 = \frac{1}{2} \\ 4b_2 + 2b_1 = 0 & \Longrightarrow b_1 = -1 \\ 2b_2 + 2b_1 + 2b_0 = 0 & \Longrightarrow b_0 = \frac{1}{2} \end{cases}$

$$\Longrightarrow x_p(t) = \frac{1}{2} - t + \frac{1}{2}t^2 \qquad \text{– particular solution tp [C]}$$

Then, the general solution to [C] is

$$x(t) = e^{-t}(C_1\cos(t) + C_2\sin(t)) + \frac{1}{2} - t + \frac{1}{2}t^2 \tag{4.21}$$

Now suppose we have initial conditions $\begin{cases} x(0) = \frac{1}{2} \\ x'(0) = 0 \end{cases}$

Notice that $x'(t) = -e^{-t}(C_1\cos(t) + C_2\sin(t)) + e^{-t}(-C_1\sin(t) + C_2\cos(t)) - 1 + t$

$$\begin{cases} x(0) = C_1 + \frac{1}{2} = \frac{1}{2} & \Longrightarrow C_1 = 0 \\ x'(0) = -C_1 + C_2 - 1 = 0 & \Longrightarrow C_2 = 1 \end{cases} \tag{4.22}$$

Then the particular solution that satisfies $\begin{cases} x(0) = \frac{1}{2} \\ x'(0) = 0 \end{cases}$ is

$$x_p(t) = e^{-t}\sin(t) + \frac{1}{2} - t + \frac{1}{2}t^2 \tag{4.23}$$

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References