Lecture 3: Review Session #3

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

3.1 First order linear ODE

$$\begin{cases} x' + a(t)x = g(t) & \text{[C]} \\ x' + a(t)x = 0 & \text{[H]} \end{cases} \tag{3.1}$$

Integrating factor

$$x'b(t) + a(t)b(t)x = b(t)g(t) \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)b(t)) = x'(t)b(t) + x(t)b'(t) \tag{3.3}$$

We need

$$b'(t) = a(t)b(t) \Longrightarrow b(t) = C_3 e^{\int a(t)\mathrm{d}t} \tag{3.4}$$

Then $b(t) = C_3 e^{\int a(t)\mathrm{d}t}$ – integrating factor

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)b(t)) = b(t)g(t) \tag{3.5}$$

$$\Longrightarrow \quad x(t)b(t) = \int b(t)g(t)\mathrm{d}t + C_1 \tag{3.6}$$

$$\Longrightarrow \quad x(t) = \frac{1}{b(t)} \int b(t)g(t)\mathrm{d}t + \frac{C_1}{b(t)} \tag{3.7}$$

$$\Longrightarrow \quad x(t) = e^{-\int a(t)\mathrm{d}t} \int e^{\int a(t)\mathrm{d}t} g(t)\mathrm{d}t + C_1 e^{-\int a(t)\mathrm{d}t} = \text{(particular solu)} + \text{(general solu)} \tag{3.8}$$

3.2 Linear ODE

$$L(t)x(t) = g(t) \tag{3.9}$$

$$L(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} + ... + a_1(t)\frac{\mathrm{d}}{\mathrm{d}t} + a_0(t) \tag{3.10}$$

$$\Longrightarrow x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + ... + a_1(t)x'(t) + a_0(t)x(t) = g(t) \qquad \text{[C] – complete equation} \tag{3.11}$$

$$L(t)x(t) = 0 \qquad \text{[H] – homogeneous equation} \tag{3.12}$$

Remark: We will denote $\{C\}$ the set of solutions to [C], $\{H\}$ the set of solutions to [H]. ■

Theorem 3.1 *If $g, a_0, a_1, ...$ are continuous, $\exists$ a unique solution to [C] for each initial condition $x(t_0) = x_0, x'(t_0) = x'_0, ..., x^{(n-1)}(t_0) = x_0^{(n-1)}$.*

[1]Visit `http://www.luzk.net/misc` for updates.

Proof: Follows from the n-th order version of Theorem 1.1. [1] ■

Theorem 3.2 *$x(t)$ is a solution to [C] $\iff$ $x(t) = x_h(t) + x_p(t)$ for some $x_p(t)$ [particular solution to [C]], and where $x_n(t)$ is the general solution to [H].*

Proof:
$\Longleftarrow$:

$$L(t)(x_h(t) + x_p(t)) = L(t)x_h(t) + L(t)x_p(t) = g(t) + 0 = g(t) \tag{3.13}$$

$\Longrightarrow$:

$$L(t)x(t) = g(t) \ \& \ L(t)x_h(t) = 0 \Longrightarrow L(t)(x(t) - x_h(t)) = g(t) \tag{3.14}$$

Let $x_p(t) = x(t) - x_h(t)$, and we have

$$x(t) = x_h(t) + x_p(t) \tag{3.15}$$

■

Theorem 3.3 *$\{H\}$ is a vector space.*

Proof: Let $E \subseteq \mathbb{R}$ be the domain of $x_n(t)$. Then let $\mathcal{F}$ be the space of functions mapping $E \mapsto \mathbb{R}$. This is a vector space, and $\{H\} \subseteq \mathcal{F}$. Let $c_1, c_2 \in \mathbb{R}, x_1(t), x_2(t) \in \{H\}$. Then

$$L(t)[c_1x_1(t) + c_2x_2(t)] = c_1L(t)x_1(t) + c_2L(t)x_2(t) = 0 + 0 = 0. \tag{3.16}$$

Hence, $c_1x_1(t) + c_2x_2(t) \in \{H\}$. Hence, $\{H\}$ is a subspace of $\mathcal{F}$. $\Longrightarrow$ $\{H\}$ is a vector space. ■

Theorem 3.4 *Let $x_1(t), ..., x_n(t) \in \{H\}$ be the particular solutions that satisfies the following initial conditions:*

$$\begin{aligned} x_1(t_0) &= 1, x_1'(t_0) = 0, ..., x_1^{(n-1)}(t_0) = 0 & (3.17)\\ x_2(t_0) &= 0, x_2'(t_0) = 1, ..., x_2^{(n-1)}(t_0) = 0 & (3.18)\\ &\vdots \\ x_n(t_0) &= 0, x_n'(t_0) = 0, ..., x_n^{(n-1)}(t_0) = 1 & (3.19) \end{aligned}$$

Then $\{x_1(t), ..., x_n(t)\}$ is the basis of $\{H\}$.

Proof: First, $\{x_1(t), ..., x_n(t)\}$ are linearly independent. To see this, consider

$$c_1x_1(t) + ... + c_nx_n(t) = 0. \tag{3.20}$$

Evaluate it at $t = t_0$,

$$c_1x_1(t_0) + ... + c_nx_n(t_0) = 0 \Longrightarrow c_1 = 0. \tag{3.21}$$

Then differentiate (3.21) w.r.t. to t:

$$c_1x_1'(t) + ... + c_nx_n'(t) = 0, \tag{3.22}$$

and evaluate it at $t = t_0$,

$$c_1x_1'(t_0) + ... + c_nx_n'(t_0) = 0 \Longrightarrow c_2 = 0. \tag{3.23}$$

[1] Warning: The labelling of theorem is likely to be inconsistent in these notes.

Simlarly, we can get $c_3, ..., c_n = 0$. Hence, $\{x_1(t), ..., x_n(t)\}$ are linearly independent.

Now take any $z(t) \in \{H\}$. And suppose

$$z(t_0) = z_0, z'(t_0) = z_0', ..., z^{(n-1)}(t_0) = z_0^{(n-1)}.$$

Let

$$\tilde{z}(t) = z_0 x_1(t) + z_0' x_2(t) + ... + z_0^{(n-1)} x_n(t) \tag{3.24}$$

Then

$$\tilde{z}(t_0) = z_0 x_1(t_0) + z_0' x_2(t_0) + ... + z_0^{(n-1)} x_n(t_0) = z_0 * 1 + 0 + ... + 0 = z_0 \tag{3.25}$$

$$\vdots$$

$$\tilde{z}^{(n-1)}(t_0) = z_0 x_1^{(n-1)}(t_0) + z_0' x_2^{(n-1)}(t_0) + ... + z_0^{(n-1)} x_n^{(n-1)}(t_0) = z_0^{(n-1)} \tag{3.26}$$

As $L(t)\tilde{z}(t) = z_0 L(t)x_1(t) + z_0' L(t)x_2(t) + ... + z_0^{(n-1)} L(t)x_n(t) = 0$, $\tilde{z}(t)$ is a solution to [H]. Further, z and $\tilde{z}$ satisfy the same initial conditions. Hence, by uniqueness, $z = \tilde{z}$. Hence, $\{x_1(t), ..., x_n(t)\}$ is the basis of $\{H\}$. ■

Corollary 3.5 $\dim\{H\} = n$.

3.3 Linear ODE with constant coefficient

$Lx(t) = g(t)$, where $L = \frac{\mathrm{d}^n}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} + ... + a_1\frac{\mathrm{d}}{\mathrm{d}t} + a_0$

$$\begin{cases} Lx(t) = g(t) & \iff x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + ... + a_1 x'(t) + a_0 x(t) = g(t) \quad [C] \\ Lx(t) = 0 & \iff x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + ... + a_1 x'(t) + a_0 x(t) = 0 \quad [H] \end{cases} \tag{3.27}$$

Definition 3.6

$$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0 \tag{3.28}$$

is the *characteristic equation* *associated to [H].*

Theorem 3.7 $x(t) = e^{\lambda t}$ *is a solution to [H]* $\iff$ λ *is a solution to the characteristic equation associated with [H].*

Proof: $x(t) = e^{\lambda t} \Longrightarrow x'(t) = \lambda e^{\lambda t}, x''(t) = \lambda^2 e^{\lambda t}, ..., x^{(n)}(t) = \lambda^n e^{\lambda t}$ Then substitute into [H] $\Longrightarrow$

$$\lambda^n e^{\lambda t} + a_{n-1}\lambda^{n-1}e^{\lambda t} + ... + a_1\lambda e^{\lambda t} + a_0\lambda e^{\lambda t} = 0 \tag{3.29}$$

$$\iff \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0\lambda = 0 \tag{3.30}$$

■

3.4 Complex Numbers

$\mathbb{C}$ – the set of Complex numbers.

If $z \in \mathbb{C}$, then $Z = a + bi$, where a is the real part and b is the imaginary part.

Polar form: We use $|z| = \sqrt{a^2 + b^2}$ – modulus. Then $z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$.

If $\theta = \pi$, then $e^{i\pi} = -1$ (Euler's identity).

References