Econ 626: Quantitative Methods II **Fall 2018**

Lecture 2: Review Session #2

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

2.1 Reducible to homogeneous

$$M(x,t)\mathrm{d}x + N(x,t)\mathrm{d}t = 0 \quad \text{if} \quad \begin{cases} M(\lambda x, \lambda t) = \lambda^k M(x,t) \\ N(\lambda x, \lambda t) = \lambda^k N(x,t) \end{cases} \tag{2.1}$$

$$\iff \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{N(x,t)}{M(x,t)} = -\frac{(\frac{1}{t})^k N(x,t)}{(\frac{1}{t})^k M(x,t)} = -\frac{N(\frac{x}{t},1)}{M(\frac{x}{t},1)} = f(\frac{x}{t}), \tag{2.2}$$

assuming M and N are homogeneous of order k.

Example:

$$(t+x)\mathrm{d}t - (x-t)\mathrm{d}x = 0 \Longrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\frac{x}{t}+1}{\frac{x}{t}-1}$$

Set $z = \frac{x}{t}$, then $f(z) = \frac{z+1}{z-1}$.

$$\int \frac{\mathrm{d}z}{f(z)-z} = \ln|t| + C \Longrightarrow \int \frac{\mathrm{d}z}{\frac{z+1}{z-1} - z} = \ln|t| + C \Longrightarrow \cdots \Longrightarrow \ln|z^2 - 2z - 1| = -2\ln|t| + C_2$$

$$\Longrightarrow |z^2 - 2z - 1| = C_3|t|^{-2} \Longrightarrow z^2 - 2z - 1 = \frac{C_4}{t^2}$$

$$\Longrightarrow (\frac{x}{t})^2 - 2\frac{x}{t} - 1 = \frac{C_4}{t^2} \Longrightarrow x^2 - 2tx - t^2 - C_4 = 0$$

$$\Longrightarrow x = t \pm \sqrt{2t^2 + C_4}$$

■

2.2 Exact ODE

$$M(x,t)\mathrm{d}x + N(x,t)\mathrm{d}t = 0 \tag{2.3}$$

Remember:

$$\mathrm{d}f(x,t) = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial t}\mathrm{d}t$$

If $f = 0$, then $\mathrm{d}f = 0$. An implication is that

$$\frac{\partial M}{\partial t} = \frac{\partial N}{\partial x}$$

[1] Visit `http://www.luzk.net/misc` for updates.

Then $\frac{\partial f}{\partial x} = M(x,t)$ for some $f(x,t) = c$,

$$\int \partial f = \int M(x,t)\partial x$$

Example: $(t+x)\mathrm{d}t - (x-t)\mathrm{d}x = 0$.

$\dfrac{\partial M}{\partial t} = 1 = \dfrac{\partial N}{\partial x}$ is satisfied.

$$\frac{\partial f}{\partial t} = N(x,t) = t + x$$

$$\int \partial f = \int (t+x)\partial t \iff f(x,t) = \frac{t^2}{2} + tx + g(x)$$

$$\implies \frac{\partial f}{\partial x} = t + g'(x) = t - x \implies g'(x) = -x \implies g(x) = -\frac{x^2}{2} + C_1$$

$$\implies C = f(x,t) = \frac{t^2}{2} + tx - \frac{x^2}{2} + C_1$$

$$\implies x^2 - 2tx - t^2 + C_2 = 0$$

$$\implies x(t) =$$

■

2.3 Inexact ODE

$$M(x,t)\mathrm{d}x + N(x,t)\mathrm{d}t = 0 \tag{2.4}$$

But $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial x}$ is not satisfied.

Multiplying by $\mu(x,t)$

$$\implies \mu(x,t)M(x,t)\mathrm{d}x + \mu(x,t)N(x,t)\mathrm{d}t = 0 \tag{2.5}$$

$$\tilde{M}(x,t) = \mu(x,t)M(x,t), \tilde{N}(x,t) = \mu(x,t)N(x,t)$$

Want to find $\mu(x,t)$, such that

$$\frac{\partial \tilde{M}}{\partial t} = \frac{\partial \tilde{N}}{\partial x}.$$

which implies

$$\frac{\partial \mu}{\partial t}M + \mu\frac{\partial M}{\partial t} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

which is a PDE.

Try $\mu(x)$ or $\mu(t)$

Example:

$$2t\mathrm{d}x + x\mathrm{d}t = 0$$

Conjecture $\mu(x)$ (not a function of t)

$$\mu(x)2t\mathrm{d}x + \mu(x)x\mathrm{d}t = 0 \implies \mu(x) = C_1 x$$

Set $C_1 = 1$ w.l.o.g.

$$\implies 2tx\mathrm{d}x + x^2\mathrm{d}t = 0$$

then our previous method could apply. (Exercise) ■

2.4 Bernoulli ODE

$$
\begin{aligned}
& x' + P(t)x = Q(t)x^n && (2.6)\\
\Longrightarrow\quad & x'x^{-n} + P(t)x^{1-n} = Q(t) && (2.7)\\
\Longleftrightarrow\quad & \frac{1}{1-n}\frac{\mathrm{d}x^{1-n}}{\mathrm{d}t} + P(t)x^{1-n} = Q(t) && (2.8)\\
& \text{Let}\quad z = x^{1-n} && (2.9)\\
\Longrightarrow\quad & \frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}t} + P(t)z = Q(t) && (2.10)\\
\Longrightarrow\quad & \frac{\mathrm{d}z}{\mathrm{d}t} + (1-n)P(t)z = (1-n)Q(t) && (2.11)
\end{aligned}
$$

which is a first order linear ODE.

2.5 First order linear ODE

$Lx(t) = g(t)$, where $L = a_0(t) + a_1(t)\frac{\mathrm{d}}{\mathrm{d}t}$

$$\Longrightarrow a_0(t)x(t) + a_1(t)x'(t) = g(t)$$

WLOG, set $a_1(t) = 1$, then we get

$$x'(t) + a(t)x(t) = g(t) \qquad \text{[C] – the complete equation} \tag{2.12}$$

Variabtion of parameters (constants)/Lagrangian method

Start with the homogeneous equation [H]

$$x'(t) + a(t)x(t) = 0 \qquad \text{[H]} \tag{2.13}$$

$$\Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)x \tag{2.14}$$

$$\vdots$$

$$\Longrightarrow \quad x(t) = C_3 e^{-\int a(t)\mathrm{d}t} \tag{2.15}$$

which is the general solution to [H].

Then we look for a particular solution to [C]. Guess

$$x(t) = C(t)e^{-\int a(t)\mathrm{d}t} \tag{2.16}$$

and substitute into [C], we get

$$C'(t)e^{-\int a(t)\mathrm{d}t} + C(t)e^{-\int a(t)\mathrm{d}t}(-a(t)) + a(t)x(t) = g(t) \tag{2.17}$$

$$C'(t)e^{-\int a(t)\mathrm{d}t} = g(t) \tag{2.18}$$

Hence, we have

$$C'(t) = g(t)e^{\int a(t)\mathrm{d}t} \tag{2.19}$$

$$\Longrightarrow \int \mathrm{d}C = \int g(t)e^{\int a(t)\mathrm{d}t}\mathrm{d}t \tag{2.20}$$

$$C(t) = \int g(t)e^{\int a(t)\mathrm{d}t}\mathrm{d}t + C_1 \tag{2.21}$$

$$\begin{aligned}\Longrightarrow x(t) &= \left(\int g(t)e^{\int a(t)\mathrm{d}t}\mathrm{d}t + C_1\right)e^{-\int a(t)\mathrm{d}t} \\ &= C_1 e^{-\int a(t)\mathrm{d}t} + e^{-\int a(t)\mathrm{d}t}\int g(t)e^{\int a(t)\mathrm{d}t}\mathrm{d}t\end{aligned} \tag{2.22}$$

Note that $C_1 e^{-\int a(t)\mathrm{d}t}$ is a general solution to [H] and $\tilde{C}(t)e^{-\int a(t)\mathrm{d}t}$ with $\tilde{C}(t) = \int g(t)e^{\int a(t)\mathrm{d}t}\mathrm{d}t$ is a particular solution to [C]. Combined together, they form the general solution.

References