Econ 626: Quantitative Methods II **Fall 2018**

Lecture 1: Review Session #1

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

Differential Equations

1.1 Basic concepts

Definition 1.1 (ODE) *Let $E \in \mathbb{R}$ and $x \in E \to \mathbb{R}$ be an unknow function. An ODE of order n is an equation of the from*

$$F(t, x, x', x'', ..., x^{(n)}) = 0$$

where $F(\cdot)$ is known, real-valued.

This is an implicit equation. We will work with the explicit equations:

$$x^{(n)} = f(t, x, x', x'', ..., x^{(n-1)})$$

$f(\cdot)$ is an known real-valued function.

Definition 1.2 (PDE) *Let $E \in \mathbb{R}^k$. An PDE of order n is an equation of the from*

$$F(t, x, \mathrm{D}x, \mathrm{D}^2 x, ..., \mathrm{D}^n x) = 0$$

where $F(\cdot)$ is known, real-valued.

Example: (ODE)

$$[x'''(t)]^4 + e^{-\xi t} x''(t) + x(t) = \tan t$$

Here, the order is 3, the degree is 4., exogenous variable: t, endogenous variable: x. ■

Theorem 1.3 *Let $E \in \mathbb{R}^2$, and $f : E \to \mathbb{R}$. If f is continuously differentiable (C^1) at $(t_0, x_0) \in E$, then $\exists \epsilon > 0$ and a unique C^1 function $t \mapsto x(t)$, such that*

$$x'(t) = f(t, x(t)), \forall t \in (t_0 - \epsilon, t_0 + \epsilon),$$

and also $x(t_0) = x_0$

Remark 1.4 *A general solution to an ODE is a set of all solutions. A particular solutions is a solution that satisfies initial conditions.*

The number of initial conditions must be equal to the order of an ODE. For example, if you solve

$$x^{(n)}(t) = f(t, x', ..., x^{(n-1)})$$

provide $x(t_0), x'(t_0), ..., x^{(n-1)}(t_0)$.

Definition 1.5 *An ODE is linear if it takes the form* $\mathrm{L}x(t) = g(t)$*, where* g *is a known real-valued function,* L *is the linear differential operator,*

$$\mathrm{L} = a_0(t) + a_1(t)\frac{\mathrm{d}}{\mathrm{d}t} + ... + a_n(t)\frac{\mathrm{d}^n}{\mathrm{d}t^n}.$$

Then

$$\mathrm{L}x(t) = a_0(t)x(t) + a_1(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + ... + a_n(t)\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} = g(t)$$

Definition 1.6 *An ODE is nonlinear if it is not linear.*

1.2 Some common types of ODE

1.2.1 Separable ODEs

$$\begin{aligned} x'(t) = f(x)g(t) \Longrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = f(x)g(t) \Longrightarrow \frac{\mathrm{d}x}{f(x)} = g(t)\mathrm{d}t \\ \Longrightarrow \int \frac{\mathrm{d}x}{f(x)} = \int g(t)\mathrm{d}t \end{aligned} \tag{1.1}$$

Example:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{t} \iff \int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}t}{t} \Longrightarrow \ln|x| = \ln|t| + C_1 \Longrightarrow |x| = |t|e^{C_1} \equiv C_2|t| \\ \Longrightarrow x = C_3 t \end{aligned} \tag{1.2}$$

Suppose $x(1) = 5$, then $|x| = C_2|t| \Longrightarrow x(t) = 5t$. ■

1.2.2 Reducible to separable

Suppose we have

$$x'(t) = f(ax + bt + c)$$

Let $z = ax + bt + c$, then $z'(t) = ax'(t) + b = f(z) + b$, then $\frac{\mathrm{d}z}{af(z)+b} = \mathrm{d}t$,

$$\int \frac{\mathrm{d}z}{af(z) + b} = \int \mathrm{d}t = t + C_1 \Longrightarrow \text{solve for } z(t) \Longrightarrow \text{solve for } x(t)$$

Example: Let $a = -1, b = 1, c = 0$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t - x} + 1$$

The left hand side seems not separable. Let $z = t - x$, so $f(z) = \frac{1}{z} + 1$. Then

$$\int \frac{\mathrm{d}z}{-\frac{1}{z} - 1 + 1} = t + C_1 \iff \int z\mathrm{d}z = -t - C_1 \Longrightarrow \frac{z^2}{2} = -t + C_2 \iff z^2 = -2t + C_3$$

$$x(t) = t \pm \sqrt{-2t + C_3} \quad \text{- general solution} \tag{1.3}$$

With initial condition $x(0) = 5$, then $5 = \pm\sqrt{C_3} \Longrightarrow C_3 = 25$. Hence

$$x(t) = t + \sqrt{-2t + 25} \quad \text{- particular solution} \tag{1.4}$$

■

1.2.3 Homogeneous ODEs

Homogeneous ODEs have the following form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(\frac{x}{t})$$

Let $z = \frac{x}{t} \Longrightarrow x(t) = tz(t), x'(t) = z(t) + tz'(t)$. Hence, the original ODE can be transformed into

$$z(t) + tz'(t) = f(z) \Longrightarrow \frac{\mathrm{d}z}{f(z) - z} = \frac{\mathrm{d}t}{t} \tag{1.5}$$

$$\int \frac{\mathrm{d}z}{f(z) - z} = \ln|t| + C \tag{1.6}$$

Example: $\frac{\mathrm{d}x}{\mathrm{d}t} = \tan(\frac{x}{t}) + \frac{x}{t}$. So $f(z) = \tan z + z$. Hence

$$\int \frac{\mathrm{d}z}{\tan z} = \ln|t| + C \Longrightarrow \ln|\sin z| = \ln|t| + C \iff |\sin z| = C_1|t| \iff z(t) = \arcsin(C_2 t) \tag{1.7}$$

$$\Longrightarrow \frac{x(t)}{t} = \arcsin(C_2 t) \Longrightarrow x(t) = t\arcsin(C_2 t) \quad \text{-} \quad \text{general solution} \tag{1.8}$$

Now suppose $x(t_0) = x_0$, ...,

$$\Longrightarrow x(t) = t\arcsin(\frac{t}{t_0}\sin(\frac{x_0}{t_0})) \tag{1.9}$$

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References