Lecture 8: Continue on Dynamic Programming

Lecturer: Prof. Daniel Levy *Scribes: Zhikun Lu*

Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

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Metrics on a set of functions

Let X be a set of continuous functions that map $[a, b] \to \mathbb{R}$. Then, $\forall f, g \in X$, we can define

1. Sup-metric
$$d_{sup}(f, g) \equiv \sup_{x \in [a,b]} |f(x) - g(x)| \tag{8.1}$$

2. L^2-metric
$$d_2(f, g) \equiv \left(\int_a^b [f(x) - g(x)]^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \tag{8.2}$$

Lemma 8.1 *Let C be a set of continuous real-valued functions defined on $[a, b] \subset \mathbb{R}$. Prove that (C, d) is a metric space if $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$.*

Proof:

1. $d(f, g) = \sup |...| \geq 0$ by properties of absolute value.
2. $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| \iff f(x) = g(x), \forall x \in [a, b]$.
3. $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| = \sup_{x \in [a,b]} g(x) - f(x)| = d(g, f)$
4. $$\begin{aligned} d(f, h) &= \sup_{x \in [a,b]} |f(x) - h(x)| \\ &= \sup_{x \in [a,b]} |f(x) - g(x) + g(x) - h(x)| \\ &\leq \sup_{x \in [a,b]} \Big\{ |f(x) - g(x)| + |g(x) - h(x)| \Big\} \\ &\leq \sup_{x \in [a,b]} |f(x) - g(x)| + \sup_{x \in [a,b]} |g(x) - h(x)| \\ &= d(f, g) + d(g, h) \end{aligned}$$

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Theorem 8.2 (Banach space theorem, a version of SLP pp.47-49) *Let $B(X) \subseteq F(X)$ be the set of bounded functionals that map from X to $\mathbb{R}$. Then:*

[1] Visit `http://www.luzk.net/misc` for updates.

1. *$||f|| = \sup_{x \in X} |f(x)|$ is a norm on $B(X)$.*
2. *$B(X)$ is a normed linear space.*
3. *$B(X)$ is a Banach space.*

Proof:

1. From the def of $||f|| = \sup_{x \in X} |f(x)|$ it follows that

   a) $||f|| \geq 0$ be properties of absolute value.
   b) $||f|| = 0$ if f is the zero functional: $f(x) = 0 \ \forall x$
   c) $||\alpha f|| = |\alpha| ||f||$ because

   $$||\alpha f|| = \sup |\alpha f(x)| = \sup\{|\alpha||f(x)|\} = |a| \sup |f(x)|$$

   d) $||f+g|| = \sup_{x \in X} |(f+g)(x)| = \sup |f(x)+g(x)| \leq \sup\{|f(x)|+|g(x)|\} \leq \sup\{|f(x)|\}+\sup\{|g(x)|\} = ||f|| + ||g||$

2. $\forall f \in B(X)$ and $\forall \alpha \in \mathbb{R}$,

   $$\alpha f(x) \leq |\alpha| ||f(x)|| \ \forall x \in X \Longrightarrow \alpha f(x) \in B(X) \ \forall \alpha \in \mathbb{R}$$

   $\forall f, g \in B(X)$, $(f+g)(x) = f(x) + g(x) \leq ||f(x)|| + ||g(x)|| \Longrightarrow B(X)$ is closed under addition and scalar multipliation, i.e., $B(X)$ is a subspace of $F(X)$, i.e., $B(X)$ is a normed linear space.

3. To show that $B(X)$ is a Banach space, we need to show that $B(X)$ is complete.

   Assume that $\{f_n\}$ is a Cauchy sequence in $B(X)$.

   Note: Another way of defining a Cauchy sequence is as follows: A sequence in a metric space (X, d) is Cauchy if $\forall n, m \in \mathbb{N}$, $\lim_{\substack{n \to \infty \\ m \to \infty}} d(x_n, x_m) = 0$.

   Then, $\forall x \in X$,

   $$|f_n(x) - f_m(x)| \leq ||f_n - f_m|| \longrightarrow 0$$

   because the RHS is the supremum, and because $f_n(x)$ is Cauchy sequence.

   Therefore, $\forall x \in X$, $f_n(x)$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, this sequence converges: $f(x) \equiv \lim_{n \to \infty} f_n(x)$.

   To show that $||f_n - f|| \to 0$:

   Since f_n is Cauchy, let $\epsilon > 0$ and $N \in \mathbb{N}$ s.t.

   $$||f_n - f_m|| < \frac{\epsilon}{2}, \ \forall m, n > N$$

   Then $\forall x \in X$ and $\forall n > N$,

   $$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq ||f_n - f_m|| + ||f_m - f||$$

   By choosing m (which might depend on x) suitably, each term on RHS can be made smaller than $\frac{\epsilon}{2}$ and therefore $|f_n(x) - f(x)| < \epsilon \ \forall x \in X$ and $\forall n > N$. Since this ($< \epsilon$) is true for all $x \in X \ \forall n > N$, it will be true for the sup as well.

   $\Longrightarrow ||f_n - f|| = \sup |f_n(x) - f(x)| < \epsilon$.

   $\Longrightarrow f = \lim_{n \to \infty} f_n$. Obivously, $||f||$ is finite.

   $\Longrightarrow f \in B(X)$.

■

8.1 Contraction Mappting

Let (S, ρ) be a metric space and $T : S \to S$ be operator. T is a contraction mapping with modulus β if for some $\beta \in (0, 1)$, $\rho(T(x), T(y) \leq \beta\rho(x, y),\ \forall x, y \in S$.

Contraction coefficient is the infimum of all these β's.

Example: $S = [a, b] \subset \mathbb{R}$, $\rho(x, y) = |x - y|$ ■

Theorem 8.3 *$T : [a, b] \to [a, b]$ is a contraction if for some $\beta \in (0, 1)$,*

$$\frac{T(x) - T(y)}{|x - y|} \leq \beta < 1, \quad \forall x, y \in [a, b],\ x \neq y$$

Example: $T(x) = \frac{x}{2}$

$$\rho(T(x), T(y)) = \rho\left(\frac{x}{2}, \frac{y}{2}\right) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| = \frac{1}{2}\rho(x, y)$$

$\Longrightarrow T$ is a contraction mapping with $\beta = \frac{1}{2}$.

`[Insert a graph here]`

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Theorem 8.4 (Uniform Continuity of Contraction Mappting) *If $T : S \to S$ is a contraction in a metric space (S, d), then it is uniformly continuous.*

Proof: $T : S \to S$ is uniformly continuous if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$, s.t. $\forall x, y \in S$, $d(x, y) < \delta \Longrightarrow d(T(x), T(y)) < \epsilon$.

Since T is contraction, we know that $\exists \beta \in (0, 1)$ s.t. $d(Tx, Ty) \leq \beta d(x, y),\ \forall x, y \in S, x \neq y$. Let $\epsilon > 0$ and assume that $d(x, y) < \delta$ where $\delta = \frac{\epsilon}{\beta}$. Then:

$$d(Tx, Ty) \leq \beta d(x, y) < \beta\delta = \beta\frac{\epsilon}{\beta} = \epsilon$$

We showed that $d(x, y) < \delta \Longrightarrow d(Tx, Ty) < \epsilon, \forall \epsilon > 0$, which prove that T is uniformly continuous. ■

8.2 Fixed Point of a Contraction

According to the diagram, it is clear that a contraction mapping will have a fixed point which is unique. That is, $\exists v^* \in [a, b]$ s.t. $T(v^*) = v^*$.

`[Insert a graph here]`

Algorithm for finding v^*

Consider following sequence $\{v_n\}_{n=0}^{\infty}$, where v_0 is some initial value $v_0 \in [a, b]$:

$$\begin{aligned} v_1 &= T(v_0) \\ v_2 &= T(v_1) = T(T(v_0)) = T^2(v_0) \\ v_3 &= T(v_2) = T(T^2(v_0)) = T^3(v_0) \\ &\vdots \\ v_n &= T(v_{n-1}) = T^n(v_0) \end{aligned}$$

Theorem 8.5 (Contraction Mapping (Banach Fixed Point Theorem)) *If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction with modulus β, then*

a) T has a unique fixed point v^* in S.

b) $\{v_n(v_0)\}_{n=1}^{\infty} \to v^*$ $\forall v_0 \in S$.

Proof: First, we need to show that $\{v_n(v_0)\}_{n=1}^{\infty}$ is Cauchy.

Let $v_0 \in S$ be an arbitary initial point in S and form the sequence $\{v_n(v_0)\}_{n=1}^{\infty}$ by using the recursion on T:

$$\begin{aligned} v_1 &= T(v_0) \\ v_2 &= T(v_1) = T^2(v_0) \\ &\vdots \\ v_n &= T(v_{n-1}) = T^n(v_0) \\ v_{n+1} &= T(v_n) = T^{n+1}(v_0) \end{aligned}$$

Since T is a contraction,

$$\rho(v_2, v_1) = \rho(T(v_1), T(v_0)) \leq \beta\rho(v_1, v_0)$$

i.e., the distance between two successive terms of $\{v_n\}$ is bounded and decreasing in n.

$$\rho(v_{n+1}, v_n) = \rho(T(v_n), T(v_{n-1})) \leq \beta\rho(v_n, v_{n-1}) \leq ... \leq \beta^n \rho(v_1, v_0), \quad n = 1, 2, ...$$

To show that $\{v_n\}$ is Cauchy, consider for any $m > n$:

$$\begin{aligned} \rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_n) \\ &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_{m-2}) + \rho(v_{m-2}, v_n) \\ &\vdots \\ &\leq \underbrace{\rho(v_m, v_{m-1})}_{\leq} + \underbrace{\rho(v_{m-1}, v_{m-2})}_{\leq} + ... + \underbrace{\rho(v_{n+2}, v_{n+1})}_{\leq} + \underbrace{\rho(v_{n+1}, v_n)}_{\leq} \\ &\leq \beta^{m-1}\rho(v_1, v_0) + \beta^{m-2}\rho(v_1, v_0) + ... + \beta^{n+1}\rho(v_1, v_0) + \beta^n\rho(v_1, v_0) \\ &= \beta^n \sum_{i=0}^{m-n-1} \beta^i \rho(v_1, v_0) \\ &< \beta^n \sum_{i=0}^{\infty} \beta^i \rho(v_1, v_0) \\ &= \frac{\beta^n}{1-\beta}\rho(v_1, v_0) \\ \implies \rho(v_m, v_n) &< \frac{\beta^n}{1-\beta}\rho(v_1, v_0) \end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{\beta^n}{1-\beta} = 0$$

For $\epsilon > 0$, choose sufficiently large n and $m(> n)$ such that $\rho(v_m, v_n) < \epsilon \Longrightarrow \{v_n(v_0)\}_{n=1}^{\infty}$ is Cauchy.

Next, show that $v_n \Longrightarrow v^* \in S$

Since (S, ρ) is complete and since $\{v_n\}$ is Cauchy,

$$\lim_{n\to\infty} v_n = v^* \in S$$

Now show that v^* is a fixed point.

Since $\lim_{n\to\infty} v_n = v^*$,

$$T(v^*) = T(\lim_{n\to\infty} v_n) = \lim_{n\to\infty} T(v_n) = v^*$$

(By **Theorem 8.4**, T is (uniformly) continuous.)

$\Longrightarrow v^*$ is a fixed point of $T(v)$.

Finally, show that v^* is unique.

Suppose that v^* and v^{**} are two different fixed points, i.e. $T(v^*) = v^*$ and $T(v^{**}) = v^{**}$.

Since T is a contraction, $\exists \beta \in (0, 1)$ s.t.

$$0 < \alpha \equiv \rho(v^*, v^{**}) = \rho(T(v^*,)T(v^{**})) \le \beta\rho(v^*, v^{**}) = \beta\alpha$$

$\Longrightarrow$

$$\alpha \le \beta\alpha \quad \text{for } \alpha > 0,\ 0 < \beta < 1$$

$\Longrightarrow \alpha = 0 \Longrightarrow \rho(v^*, v^{**}) = 0 \Longrightarrow v^* = v^{**}$ ■

Theorem 8.6 (Blackwell's sufficient condition for a contraction) *Let $X \subseteq \mathbb{R}$ and let $B(X)$ be a space of bounded functions. $f : B(X) \to \mathbb{R}$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying two conditions*

1. *Monotonicity*

   $\forall f, g \in B(X)$ and $f(x) \le g(x)$ for all x,

   $$T(f)(x) \le T(g)(x)\ \forall x \in X$$

2. *Discounting*

   $\exists \beta \in (0, 1)$ s.t.

   $$T(f + \alpha)(x) \le T(f)(x) + \beta\alpha$$

   $\forall f \in B(X),\ \forall \alpha \in \mathbb{R}_+,\ \forall x \in X.$

Then T is a contraction mapping with modulus β.

Proof: For any $f, g \in B(X)$,

$$f = f + g - g = g + (f - g) \le g + ||f - g||$$

Then

$$T(f)(x) \leq T(g + ||f-g||)(x) \leq T(g)(x) + \beta||f-g||$$

$\Longrightarrow T(f)(x) - T(g)(x) \leq \beta||f-g||$

By symmetry, we also have

$$T(g)(x) - T(f)(x) \leq \beta||f-g||, \quad \text{for all } x$$

$$\Longrightarrow ||T(f) - T(g)|| \leq \beta||f-g||$$

Hence T is a contraction mapping. ■

Pseudocontraction

$T : X \to X$ in a metric space (S, ρ) is called a pseudocontraction mapping if $\forall x, y \in X,\ x \neq y$,

$$d(T(x), T(y)) < d(x, y).$$

Note: Every contraction is a pseudocontraction.

Correspondence

`[Insert a graph here]`

- Relation: Not all points in X are related to points in Y
- Function: Every point in X is related to a single point in Y.
- Correspondence:Every x is related to some points (a set) in Y.

Lemma 8.7 *Pseudocontraction has at most one fixed point.*

Definition 8.8 *A correspondence $\phi : X \rightrightarrows Y$ is a rule that assigns to every element $x \in X$ a non-empty subset $\phi(x) \subseteq Y$.*

`[Insert four graphs here]`

References