Econ 626: Quantitative Methods II	**Fall 2018**

Lecture 7: Back to Dynamic Programming

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

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7.1 Stochastic Model

$$\max_{\{C_t, K_{t+1}\}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t) \right] \tag{7.1}$$

$$\text{s.t.} \quad C_t + K_{t+1} = Z_t f(K_t) \tag{7.2}$$

where $\mathbb{E}_0 = E\{\cdot|\Omega_0\}$. Optimal choices here are contingency plans.

$\{K_t, Z_t\}$ are state variables. K_{t+1} and C_t will be a function of K_t and Z_t. Since Z is a r.v., K_{t+1} and C_t will also be r.v.'s.

Bellman's Equation

$$V(K_t, Z_t) = \begin{cases} \max \ [u(C_t) + \beta \mathbb{E}_t V(K_{t+1}, Z_{t+1})] \\ \text{s.t.} \quad C_t + K_{t+1} = Z_t f(K_t) \end{cases} \tag{7.3}$$

i.e.

$$V(K_t, Z_t) = \begin{cases} \max \ [u(C_t) + \beta \int V(K_{t+1}, Z_{t+1}) h(Z_{t+1}) \mathrm{d}Z_{t+1}] \\ \text{s.t.} \quad C_t + K_{t+1} = Z_t f(K_t) \end{cases} \tag{7.4}$$

where $h(\cdot)$ is the conditional p.d.f. of Z_{t+1}, given that the p.d.f. conditonal on Ω_t exists.

Lagrangian

$$V(K_t, Z_t) = u(C_t) + \beta \mathbb{E}_t V(K_{t+1}, Z_{t+1}) + \lambda_t [Z_t f(K_t) - C_t - K_{t+1}] \tag{7.5}$$

FONC

$$\mathbb{E}_t [u'(C_t) - \lambda_t] = 0 \Longrightarrow \lambda_t = u'(C_t) \tag{7.6}$$

$$\mathbb{E}_t [\beta V'_{K_{t+1}}(K_{t+1}, Z_{t+1}) - \lambda_t] = 0 \Longrightarrow \lambda_t = \beta \mathbb{E}_t V'_{K_{t+1}}(K_{t+1}, Z_{t+1}) \tag{7.7}$$

$\Longrightarrow$

$$u'(C_t) = \beta \mathbb{E}_t V'_{K_{t+1}}(K_{t+1}, Z_{t+1}) \tag{7.8}$$

Note:

$$\Omega_t = \{K_{t-j}, C_{t-j}, Z_{t-j} | j = 0, 1, ...\} \tag{7.9}$$

Based on the B-S Theorem, we can write

$$V'_{K_t}(K_t, Z_t) = \lambda_t Z_t f'(K_t) = u'(C_t) Z_t f'(K_t) \tag{7.10}$$

[1] Visit `http://www.luzk.net/misc` for updates.

Lead (7.10) $\Longrightarrow$

$$u'(C_t) = \beta \mathbb{E}_t u'(C_{t+1}) Z_{t+1} f'(K_{t+1}) \tag{7.11}$$

<u>Assumption</u>

$$\begin{aligned} f(K_t) &= K_t^\alpha && (7.12) \\ u(C_t) &= \ln C_t && (7.13) \end{aligned}$$

$\Longrightarrow$

$$\frac{1}{C_t} = \beta \mathbb{E}_t \frac{1}{C_{t+1}} Z_{t+1} \alpha (K_{t+1})^{\alpha-1} \tag{7.14}$$

Guess

$$\begin{aligned} K_{t+1} &= \theta Z_t K_t^\alpha && (7.15) \\ C_t &= (1-\theta) Z_t K_t^\alpha && (7.16) \end{aligned}$$

(7.14) and (7.16) $\Longrightarrow$

$$\begin{aligned} \frac{1}{(1-\theta)Z_t K_t^\alpha} &= \beta \mathbb{E}_t \frac{1}{(1-\theta)\cancel{Z_{t+1}} K_{t+1}^\alpha} \cancel{Z_{t+1}} \alpha (K_{t+1})^{\alpha-1} && (7.17) \\ \frac{1}{(1-\theta)Z_t K_t^\alpha} &= \alpha\beta \mathbb{E}_t \frac{1}{(1-\theta)K_{t+1}} && (7.18) \\ \frac{1}{\cancel{(1-\theta)Z_t K_t^\alpha}} &= \alpha\beta \mathbb{E}_t \frac{1}{\cancel{(1-\theta)}\theta \cancel{Z_t K_t^\alpha}} && (7.19) \\ 1 &= \alpha\beta \frac{1}{\theta} && (7.20) \\ \Longrightarrow \quad \theta &= \alpha\beta && (7.21) \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} K_{t+1}^* &= \alpha\beta Z_t K_t^\alpha && (7.22) \\ C_t^* &= (1-\alpha\beta) Z_t K_t^\alpha && (7.23) \end{aligned}$$

To say something about the properties of K_{t+1}^* and C_t^*, we need to know the properties of Z_t.

Example:

$$\ln Z_t \sim N(\mu, \sigma^2) \Longrightarrow Z_t \sim LN \tag{7.24}$$

<u>Recall</u> The MGF of $x \sim N(\mu, \sigma^2)$ is given by

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \tag{7.25}$$

Since $\ln Z_t \sim N(\mu, \sigma^2)$, PDF of $\ln Z_t$ is given by

$$f(\ln Z_t) = \frac{1}{Z_t \sigma \sqrt{2\pi}} e^{-\frac{(\ln Z_t - \mu)^2}{2\sigma^2}} \tag{7.26}$$

<u>Recall</u>

$$M_y(t) = \mathbb{E}\left[e^{ty}\right] = \mathbb{E}\left[e^{t \ln Z}\right] = \mathbb{E}\left[Z^t\right] \tag{7.27}$$

$$M_{\ln Z}(t) = \mathbb{E}\left[Z^t\right] \tag{7.28}$$

$$\mathbb{E}\left[Z\right] = M_{\ln Z}(1) = e^{\mu + \frac{\sigma^2}{2}} \tag{7.29}$$

$$Var(Z) = \mathbb{E}\left[Z^2\right] - (\mathbb{E}\left[Z\right])^2 = M_{\ln Z}(2) - (M_{\ln Z}(1))^2 = e^{2\mu+2\sigma^2} - (e^{\mu+\frac{\sigma^2}{2}})^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \tag{7.30}$$

Since Z_t is log-normal, so are K^*_{t+1} and C^*_t.

$$\begin{aligned}
K_{t+1} &= \alpha\beta Z_t K_t^\alpha & (7.31)\\
\ln K_{t+1} &= \ln\alpha\beta + \ln Z_t + \alpha \ln K_t & (7.32)\\
\ln K_t &= \ln\alpha\beta + \ln Z_{t-1} + \alpha \ln K_{t-1} & (7.33)\\
\ln K_{t-1} &= \ln\alpha\beta + \ln Z_{t-2} + \alpha \ln K_{t-2} & (7.34)\\
\ln K_t &= \ln\alpha\beta + \ln Z_{t-1} + \alpha(\ln\alpha\beta + \ln Z_{t-2} + \alpha \ln K_{t-2}) & (7.35)\\
&= (1+\alpha)\ln\alpha\beta + \ln Z_{t-1} + \alpha \ln Z_{t-2} + \alpha^2 \ln K_{t-2} & (7.36)\\
&= (1+\alpha+\alpha^2)\ln\alpha\beta + \ln Z_{t-1} + \alpha \ln Z_{t-2} + \alpha^2 \ln Z_{t-3} + \alpha^3 \ln K_{t-3} & (7.37)\\
&\vdots\\
&= \ln\alpha\beta \sum_{i=0}^{t-1}\alpha^i + \sum_{i=0}^{t-1}\alpha^i \ln Z_{t-i-1} + \alpha^t \ln K_0 & (7.38)
\end{aligned}$$

As $t \to \infty$, $\alpha^t \ln K_0 \to 0$.

$$\lim_{t\to\infty} \mathbb{E}(\ln K_t) = \frac{1}{1-\alpha}\ln\alpha\beta + \sum_{i=0}^{\infty}\alpha^i \mathbb{E}\ln Z_{t-i-1} = \frac{1}{1-\alpha}\ln\alpha\beta + \frac{1}{1-\alpha}\mu \tag{7.39}$$

$$Var[\lim_{t\to\infty}\ln K_t] = \mathbb{E}[\lim_{t\to\infty}\ln K_t]^2 - \left\{\mathbb{E}\left[\lim_{t\to\infty}\ln K_t\right]\right\}^2 = ... = \frac{\sigma^2}{1-\alpha^2} \tag{7.40}$$

■

7.2 Funtionals

Functionals: A function that maps from any set X to $\mathbb{R}$, $f : X \to \mathbb{R}$ is called a functional.

Operations on Functionals

$$(f \pm g)(x) = f(x) \pm g(x) \tag{7.41}$$

$$(\alpha f)(x) = \alpha f(x) \tag{7.42}$$

Lemma: Let X be a set and let $F(X)$ be the set of all functionals on X. Show that $F(X)$ is a linear space.

Proof: $\forall f, g \in F(X)$ and $\forall \alpha \in \mathbb{R}$ we have

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

and therefore

$$(f+g) : X \to \mathbb{R} \quad \text{and} \quad \alpha f : X \to \mathbb{R}$$

. $\Longrightarrow F(X)$ is closed under addition and scalar multiplication. ■

Note: the "zero" element in $F(X)$ is the constant function $f(x) = 0\ \forall\ x \in X$.

Definition A functional $f \in F(X)$ is bounded if $\exists k \in \mathbb{R}$ s.t. $|f(x)| \le k\ \forall x \in X$.

Definition For any set X, the $B(X)$ denote the set of all bounded functionals on X.

Note: $B(X) \subseteq F(X)$.

Continuity in Metric Space

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$. Then f is continuous at $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta(x_0, \epsilon) > 0$, s.t.

$$d_1(x, x_0) < \delta(x_0, \epsilon) \Longrightarrow d_2[f(x_0), f(x)] < \epsilon.$$

Uniform Continuity

A function $f : (X, d_1) \to (Y, d_2)$ is uniformly continuous on a subset $A \subset X$ if $\forall x, y \in X$ and $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$, independent of x and y, s.t.

$$d_1(x, y) < \delta(\epsilon) \Longrightarrow d_2(f(x), f(y)) < \epsilon.$$

Lipschitz Continuity

Let X and Y be normed vactor space. Then, a function $f : X \to Y$ is Lipschitz continuous if $\exists \beta > 0$, s.t. $\forall x, x_0 \in X$,

$$||f(x) - f(x_0)|| \leq \beta ||x - x_0||$$

Note:

$$\frac{||f(x) - f(x_0)||}{||x - x_0||} \leq \beta \tag{7.43}$$

Lemma (Preservation of Cauchy Property under Uniform Continuity)

Let $f : X \to Y$ be uniformly continuous. If $\{x_n\}$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

Proof: Let $\epsilon > 0$. By uniformly continuity, $\exists \delta > 0$ s.t. $d(f(x_n), f(x_m)) < \epsilon, \forall x_m, x_n \in X$, s.t. $d(x_m, x_n) < \delta$. Suppose that $\{x_n\}$ is Cauchy in $X \Longrightarrow \exists N \in \mathbb{N}$ s.t. $d(x_m, x_n) < \delta \forall m, n > N$. Then by uniform continuity of f, $d(f(x_m), f(x_n)) < \epsilon \; \forall m, n > N \Longrightarrow f(x_n)$ is Cauchy. ■

Theorem

A continuous function on a compact domain is uniformly continuous.

Lemma

Lipschitz continuity implies uniform continuity.

Proof: Let $f : X \to Y$ be Lipschitz continuous with modulus β. Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{2\beta}$. Then if $d(x, y) \leq \delta$, then

$$d(f(x), f(y)) \leq \beta d(x, y) \leq \beta\delta = \beta\frac{\epsilon}{2\beta} < \epsilon$$

which means f is uniformly continuous. ■

Sequences of Functions

Consider sequences $\{f_n]\}$ whose terms are real valued functions defined on a common domain $\mathbb{R}$. For each $x \in \mathbb{R}$, we can form a corresponding sequence $\{f_n(x)\}$, whose terms are the corresponding function values.

Let S be the set of $x's$ in $\mathbb{R}$ for which $\{f_n(x)\}$ converges.

Limit Function

If $\lim_{n\to\infty}\{f_n(x)\} = f(x)$, $x \in S$, then $f(x)$ is called the limit function of $\{f_n\}$ and we say that $\{f_n\}$ converges pointwisely to f on the set S.

<u>Note:</u> Suppose that $f_n(x)$ is continuous at some $x_0 \in S$, $\forall n$. Does this imply that the limit function $f(x)$ is also continuous at x_0? Not necessarily.

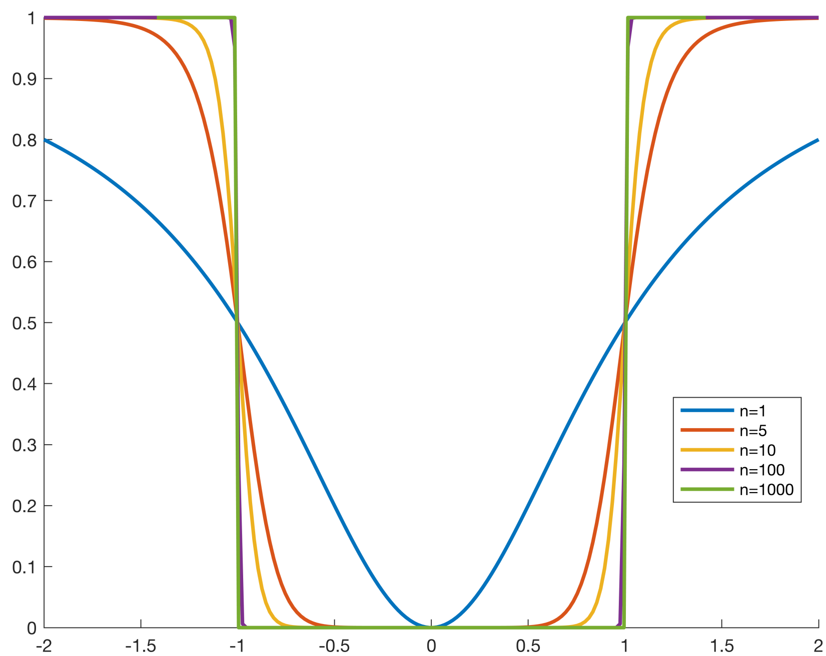
Example:

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \quad x \in \mathbb{R}, \quad n = 1, 2, ...$$

$\Longrightarrow$

$$f(x) = \begin{cases} 0, & \text{if } |x| < 1 \\ \frac{1}{2}, & \text{if } |x| = 1 \\ 1, & \text{if } |x| > 1 \end{cases}$$

■



<u>Uniform Convegence</u> of $\{f_n\}$

Let $\{f_n\}$ be a sequence of functions that converges pointwise on a set S to a limit function f, i.e., $\forall\ x \in S$ and $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$, where $N = N(x, \epsilon)$, s.t.

$$\forall\ n > N, |f_n(x) - f(x)| < \epsilon.$$

<u>Definition</u>: A sequence of functions $\{f_n\}$ is said to <u>converge uniformly</u> to f on a set S if $\forall\ \epsilon > 0$, $\exists\ N(\epsilon) \in \mathbb{N}$, s.t.

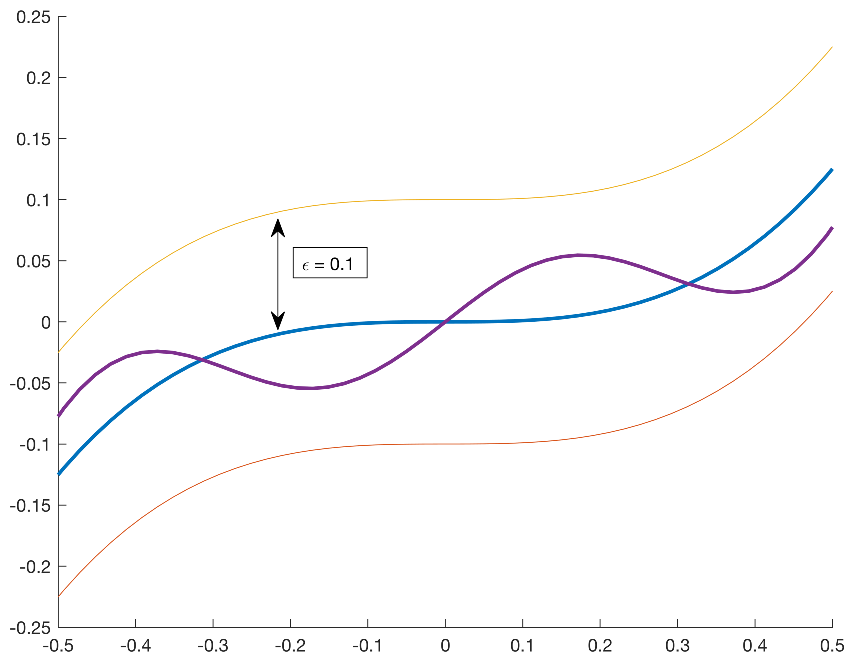
$$\forall\ n > N, |f_n(x) - f(x)| < \epsilon, \ \forall x \in S$$

<u>Geometry</u>

If f_n is real-valued $\forall n \in \mathbb{N}$, then $|f_n(x) - f(x)| < \epsilon$ mean

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon.$$

If this holds for all $n > N$ and for all $x \in S$, then $\{(x, y) \mid y = f_n(x), x \in S\}$, the entire graph, lies within the 2ϵ "bound" around f.



<u>Uniform Bounds</u> $\{f_n\}$ is <u>uniformly bounded</u> on S if $\exists M > 0$, constant, s.t. $|f_n(x)| \leq M,\ \forall x \in S$ and $\forall n$. The number M is called a <u>uniform bound</u> for f_n.

<u>Theorem</u>(Apostol)

If $f_n \to f$ uniformly on S and if each f_n is bounded on S, then f_n is uniformly bounded on S.

<u>Theorem</u> (Uniform Convergence and Continuity, Apostol)

Let $f_n \to f$ uniformly on S. If each f_n is continuous at a point $c \in S$, then the limit function f is also continuous at c.

<u>Theorem</u> (Cauchy Condition for Uniform Convergence)

Let $\{f_n\}$ be defined on S. Then there exists f s.t. $f_n \to f$ uniformly on$S$ iff $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall m, n > N$,

$$\underbrace{|f_m(x) - f_n(x)|}_{\text{Cauchy Condition}} < \epsilon, \quad \forall x \in S$$

Proof: "$\Longrightarrow$"

Assume that $f_n \to f$ uniformly on S. Let $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $n > N \Longrightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}, \forall x \in S$. Let $m > N$. Then $|f_m(x) - f(x)| < \frac{\epsilon}{2}$. Then

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

"$\Longleftarrow$"

Suppose that the Cauchy condition is satisfied $\Longrightarrow \forall x \in S, \{f_n(x)\}$ converges.

Let $f(x) = \lim_{n\to\infty} f_n(x)\ \forall\ x \in S$.

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ s.t. $\forall n > N$,

$$|f_n(x) - f_{n+k}(x)| < \frac{\epsilon}{2}, \quad \forall k = 1, 2, 3..., \quad \forall x \in S.$$

Then

$$\lim_{k\to\infty} |f_n(x) - f_{n+k}(x)| = |f_n(x) - f(x)| \leq \frac{\epsilon}{2}$$

$\Longrightarrow \forall n > N$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in S.$$

$\Longrightarrow f_n \to f$ uniformly on S. ∎

References