Econ 626: Quantitative Methods II **Fall 2018**

Lecture 6: More about Difference Equations

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

Date: September 5, 2018

6.1 First Order Linear Difference Equations

$$\underbrace{y_t}_{Endog.} = \lambda y_{t-1} + b \underbrace{x_t}_{Exog.} + a, \qquad a, b, \lambda = \text{ parameters}, \lambda \neq 1 \tag{6.1}$$

Note:

$\lambda = 1$ unit root, non-stationary, random walk

Sargent 1979, CH.9

$$y_y - \lambda y_{t-1} = bx_t + a \tag{6.2}$$

$$(1 - \lambda L)y_t = bx_t + a \tag{6.3}$$

$$y_t = \frac{1}{1 - \lambda L}(bx_t + a) + c\lambda^t \tag{6.4}$$

where $c\lambda^t$ is the "summation constant", similar to the constant in indefinite integral. We can check that

$$(1 - \lambda L)c\lambda^t = c\lambda^t - \lambda L(c\lambda^t) = c\lambda^t - \lambda c\lambda^{t-1} = 0$$

Note

$$L(x_t \pm y_t) = x_{t-1} \pm y_{t-1} \tag{6.5}$$

$$L(ax_t) = aLx_t = ax_{t-1} \tag{6.6}$$

$$y_t = (\frac{1}{1 - \lambda L})bx_t + (\frac{1}{1 - \lambda L})a + c\lambda^t \tag{6.7}$$

Assuptions

$|\lambda| < 1 \Longrightarrow La = a$.

Then

$$(\frac{1}{1 - \lambda L})a = (\frac{1}{1 - \lambda})a \tag{6.8}$$

This is because

$$(\frac{1}{1 - \lambda L})a = (\sum_{i=0}^{\infty} \lambda^i L^i)a = \sum_{i=0}^{\infty}(\lambda^i L^i a) = \sum_{i=0}^{\infty}(\lambda^i a) = a\sum_{i=0}^{\infty} \lambda^i = a(\frac{1}{1 - \lambda}) \tag{6.9}$$

[1] Visit `http://www.luzk.net/misc` for updates.

Thus,

$$\begin{aligned} y_t &= (\frac{1}{1-\lambda L})bx_t + (\frac{1}{1-\lambda L})a + c\lambda^t \\ &= b\sum_{i=0}^{\infty}\lambda^i x_{t-i} + \frac{a}{1-\lambda} + c\lambda^t \end{aligned} \tag{6.10}$$

Recall

If $|\lambda| < 1$, then $(\frac{1}{1-\lambda L})x_t = \sum_{i=0}^{\infty}\lambda^i x_{t-i}$

If $|\lambda| > 1$, then $(\frac{1}{1-\lambda L})x_t = -\sum_{i=1}^{\infty}(\frac{1}{\lambda})^i x_{t+i}$

Note: If $|\lambda| > 1$, then we would use forward-looking solution.

Example: Cagan (1956)

$$\begin{cases} m_t &= \ln M_t \\ p_t &= \ln P_t \\ p^e_{t+1} &= \ln P^e_{t+1} \end{cases} \tag{6.11}$$

Money Market: Money demand

$$\underbrace{m_t - p_t}_{\ln(\frac{M_t}{P_t})} = \alpha(\underbrace{p^e_{t+1} - p_t}_{\text{expeced inflation}}), \qquad \alpha < 0 \tag{6.12}$$

this is a(n) simplification/approximation under hyperinflation:

$$(\frac{M}{P})^d = \frac{M}{P}(i, y) = \frac{M}{P}(r + \pi^e, y) \tag{6.13}$$

When inflation is high, r and y can be treated as fixed.

Assumption

$$y = \bar{y}, r = \bar{r}$$

$$P^e_{t+1} = (1-\gamma)P_t + \gamma P_{t-1} \Longrightarrow \pi^e_{t+1} = \Gamma\pi_t, \quad \Gamma < 0 \tag{6.14}$$

which is an example of extrapolative expectation.

$$p^e_{t+1} = (1+\Gamma)p_t - \Gamma p_{t-1} \tag{6.15}$$

$$m_t - p_t = \alpha\Gamma(p_t - p_{t-1}) \tag{6.16}$$

$$\Longrightarrow p_t - \underbrace{(\frac{a\Gamma}{1+a\Gamma})}_{\lambda} p_{t-1} = \frac{1}{1+a\Gamma}m_t \tag{6.17}$$

$$p_t - \lambda p_{t-1} = \frac{1}{1+a\Gamma}m_t \tag{6.18}$$

$$p_t = \frac{1}{1+a\Gamma}(\frac{1}{1-\lambda L})m_t + c\lambda^t \tag{6.19}$$

Since $|\lambda| = |\frac{\alpha\Gamma}{1+\alpha\Gamma}| < 1$, $\Longrightarrow$

$$p_t = \frac{1}{1+a\Gamma}\sum_{i=0}^{\infty}\lambda^i m_{t-i} + c\lambda^t = \frac{1}{1+a\Gamma}\sum_{i=0}^{\infty}(\frac{\alpha\Gamma}{1+\alpha\Gamma})^i m_{t-i} + c(\frac{\alpha\Gamma}{1+\alpha\Gamma})^t \tag{6.20}$$

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Example: Perfect forsight

$$p^e_{t+1} - p_t = p_{t+1} - p_t \tag{6.21}$$

$$\pi^e_{t+1} = \pi_{t+1} \tag{6.22}$$

Then money demand is

$$m_t - p_t = \alpha(p_{t+1} - p_t) \tag{6.23}$$

$$\Longrightarrow p_{t+1} - \frac{\alpha-1}{\alpha} p_t = \frac{m_t}{\alpha} \tag{6.24}$$

$$\Longrightarrow (1 - \lambda L) p_{t+1} = \frac{m_t}{\alpha} \tag{6.25}$$

Here, $|\lambda| = |\frac{\alpha-1}{\alpha}| > 1$. Hence use backward-looking solution:

$$\begin{aligned}
p_{t+1} &= -\frac{1}{\alpha}\left(\frac{1}{1-\lambda L}\right) m_t + c\lambda^t \\
&= -\frac{1}{\alpha}\sum_{i=1}^{\infty}\left(\frac{1}{\lambda}\right)^i m_{t+i} + c\lambda^t \\
&= -\frac{1}{\alpha}\sum_{i=1}^{\infty}\left(\frac{1}{(\frac{\alpha-1}{\alpha})}\right)^i m_{t+i} + c(\frac{\alpha-1}{\alpha})^t \\
&= \frac{1-\alpha}{\alpha^2}\sum_{i=1}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^{i+1} m_{t+i} + c(\frac{\alpha-1}{\alpha})^t \\
p_t &= \frac{1-\alpha}{\alpha^2}\sum_{i=1}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^{i+1} m_{t+i-1} + c(\frac{\alpha-1}{\alpha})^{t-1}
\end{aligned} \tag{6.26}$$

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6.2 Dynamic Discrete Time Infinite Horizon Model (Ramsey)

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t), \qquad 1 > \beta > 0 \tag{6.27}$$

$$\text{s.t.} \quad c_t + B_t = (1+\rho_{t-1})B_{t-1} + y_t \tag{6.28}$$

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t u(c_t) - \sum_{t=1}^{\infty} \lambda_t[(1+\rho_{t-1})B_{t-1} + y_t - c_t - B_t] \tag{6.29}$$

Choice: $\{c_t, B_t\}_{t=1}^{\infty}$

FONC:

$$[c_t] \qquad \beta^t u'(c_t) - \lambda_t = 0 \tag{6.30}$$

$$[B_t] \qquad -\lambda_t + \lambda_{t+1}(1+\rho_t) = 0 \tag{6.31}$$

$\Longrightarrow$

$$1 + \rho_t = \frac{u'(c_t)}{\beta u'(c_{t+1})} \tag{6.32}$$

which says "objective rate of substitution" = "subjective rate of substitution".

<u>Assumption</u>: $\rho_t = \rho, \ \forall t$

$$(1-(1+\rho)L)B_t = y_t - c_t \tag{6.33}$$

$$\begin{aligned}
B_t &= \frac{1}{1-(1+\rho)L}(y_t - c_t) + d(1+\rho)^t && (6.34)\\
&= -\sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s(y_{t+s} - c_{t+s}) + d(1+\rho)^t && (6.35)\\
&= -\sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s(y_{t+s} - c_{t+s}) \qquad (d = 0 \text{ because of NPGC}) && (6.36)\\
\sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s c_{t+s} &= B_t + \sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s y_{t+s} && (6.37)
\end{aligned}$$

which is the life-time budget constaint.

<u>NPGC</u>:

$$\lim_{t\to\infty} d(1+\rho)^t = 0 \Longrightarrow d = 0 \tag{6.38}$$

<u>Assumption</u> $y_t = y, \ u(c) = \ln(c) \Longrightarrow$

$$1+\rho = \frac{1}{\beta}\frac{c_{t+1}}{c_t} \iff (1+\rho)\beta c_t = c_{t+1} \tag{6.39}$$

$$\begin{aligned}
c_{t+1} &= (1+\rho)\beta c_t && (6.40)\\
c_{t+2} &= ((1+\rho)\beta)^2 c_t && (6.41)\\
c_{t+3} &= ((1+\rho)\beta)^3 c_t && (6.42)\\
&\vdots && (6.43)\\
c_{t+s} &= ((1+\rho)\beta)^s c_t && (6.44)
\end{aligned}$$

$\Longrightarrow$

$$B_t + \sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s y = \sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s(1+\rho)^s\beta^s c_t \tag{6.45}$$

$$B_t + y\sum_{s=1}^{\infty}(\frac{1}{1+\rho})^s = c_t\sum_{s=1}^{\infty}\cancel{(\frac{1}{1+\rho})^s(1+\rho)^s}\beta^s \tag{6.46}$$

$$B_t + \frac{1}{\rho}y = \frac{\beta}{1-\beta}c_t \tag{6.47}$$

$$c_t = \frac{1-\beta}{\beta\rho}(y + \rho B_t) \tag{6.48}$$

6.3 Expectations

How to get information about expectation?

1. Survey
2. Observe behaviour
3. Assume some mechanism
   (a) Static expectation $p^e_{t+1} = p_t$
   Example: The cobweb model
   (b) Extrapolative expectations
   Example:
   $$p^e_{t+1} = \Gamma p_t + (1-\Gamma)p_{t-1} = \Gamma(p_t - p_{t-1}) + p_{t-1} \tag{6.49}$$
   Example:
   $$\begin{aligned} C_{t+1} &= \alpha + \beta y^e_{t+1} \\ &= \alpha + \beta\Gamma y_t + \beta(1-\Gamma)y_{t-1} + \epsilon_t \end{aligned} \tag{6.50}$$
   (c) Adaptive expectations

Adaptive Expectations

$$p^e_{t+1} = p^e_t + \theta(\underbrace{p_t - p^e_t}_{\text{forecast error}}), \qquad 0 < \theta < 1 \tag{6.51}$$

$$p^e_{t+1} = \theta \sum_{s=0}^{\infty} (1-\theta)^s p_{t-s} + c(1-\theta)^t \tag{6.52}$$

Rational Expectations

1) $P^e_{t+1} = \mathbb{E}[P_{t+1} \mid \Omega_t]$

2) $P^e_{t+1} - P_{t+1} = \epsilon_{t+1}$, where $\mathbb{E}(\epsilon_{t+1}) = 0$, $\text{cov}(\epsilon_t, \epsilon_{t\pm1})$.

Perfect Forsight is an extreme case of RE where $\epsilon_{t+1} = 0 \; \forall t$.

6.4 Stochastic Difference Equation

$$y_t = a \underbrace{\mathbb{E}[y_{t+1}|I_t]}_{\mathbb{E}_t y_{t+1}} + c x_t \tag{6.53}$$

Example:

$$p_t = (\frac{\alpha}{1+\alpha})\mathbb{E}[p_{t+1}|I_t] + (\frac{1}{1+\alpha})m_t \tag{6.54}$$

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Law of Iterated Expectations

$$\mathbb{E}\left[\mathbb{E}\left[X|I_{t+1}\right]|I_t\right] = \mathbb{E}\left[X|I_t\right] \tag{6.55}$$

$$
\begin{aligned}
y_t &= a\mathbb{E}_t y_{t+1} + cx_t && (6.56)\\
y_{t+1} &= a\mathbb{E}_{t+1} y_{t+2} + cx_{t+1} && (6.57)\\
\mathbb{E}_t y_{t+1} &= a\mathbb{E}_t\mathbb{E}_{t+1} y_{t+2} + c\mathbb{E}_t x_{t+1} = a\mathbb{E}_t y_{t+2} + c\mathbb{E}_t x_{t+1} && (6.58)\\
\Longrightarrow y_t &= a^2\mathbb{E}_t y_{t+2} + ac\mathbb{E}_t x_{t+1} + cx_t && (6.59)\\
y_{t+2} &= a\mathbb{E}_{t+2} y_{t+3} + cx_{t+2} && (6.60)\\
\Longrightarrow \mathbb{E}_t y_{t+2} &= a\mathbb{E}_t y_{t+3} + c\mathbb{E}_t x_{t+2} && (6.61)\\
\Longrightarrow y_t &= a^3\mathbb{E}_t y_{t+3} + a^2 c\mathbb{E}_t x_{t+2} + ac\mathbb{E}_t x_{t+1} + cx_t && (6.62)\\
&\vdots\\
y_t &= c\sum_{i=0}^{T} a^i \mathbb{E}_t x_{t+i} + a^{T+1}\mathbb{E}_t y_{t+T+1} && (6.63)
\end{aligned}
$$

Let $T \to \infty$ and assume that

$$\lim_{T\to\infty}[a^{T+1}\mathbb{E}_t y_{t+T+1}] = 0 \tag{6.64}$$

i.e. we are ruling out bubble solution.

Then we get the fundamental solution:

$$y_t = c\sum_{i=0}^{\infty} a^i \mathbb{E}_t x_{t+i} \tag{6.65}$$

6.5 Stochastic Dynamic Discrete Time Infinite Horizon Model

$$\max \quad \mathbb{E}\left[\sum_{t=1}^{\infty} \beta^t u(c_t)\right] \tag{6.66}$$

$$\text{s.t.} \quad c_t + B_t = (1+\rho_{t-1})B_{t-1} + y_t \tag{6.67}$$

Random Lagrangian Method (Kushner (1965))

FONC

$$\mathbb{E}_t[\beta^t u'(c_t) - \lambda_t] = 0 \tag{6.68}$$

$$\mathbb{E}_t[-\lambda_t + \lambda_{t+1}(1+\rho_t)] = 0 \tag{6.69}$$

Note: At time t, variables dated t and earlier are known and hence they are not R.V.'s (random variables), i.e.

$$\mathbb{E}_t x_t = \mathbb{E}[x_t|I_t] = x_t \tag{6.70}$$

$\Longrightarrow$

$$\beta^t u'(c_t) - \lambda_t = 0 \tag{6.71}$$

$$-\lambda_t + (1+\rho_t)\mathbb{E}_t\lambda_{t+1} = 0 \tag{6.72}$$

$\Longrightarrow$

$$u'(c_t) = (1+\rho_t)\beta\mathbb{E}_t u'(c_{t+1}) \tag{6.73}$$

$$\mathbb{E}_t u'(c_{t+1}) = \frac{1}{(1+\rho_t)\beta}u'(c_t) \tag{6.74}$$

$$u'(c_{t+1}) = \frac{1}{(1+\rho_t)\beta} u'(c_t) + \epsilon_{t+1} \tag{6.75}$$

with $\mathbb{E}_t \epsilon_{t+1} = 0$. Hence it's almost an AR(1) process.

Claim: If $\mathbb{E}_t(x_{t+1}) = x_t$, then $x_{t+1} = x_t + \epsilon_{t+1}$, $E_t \epsilon_{t+1} = 0$.

Comment: This is also a regression equation/model that can be used for estimation, taken to data for test directly. If $u(c) = \ln c$, then

$$\frac{1}{c_{t+1}} = \frac{1}{(1+\rho_t)\beta} \frac{1}{c_t} + \epsilon_{t+1} \tag{6.76}$$

which is a regression equation with time-varying coefficient (Kalman filter). (F. Mishikin 1986, NBER-U.C. Press)

Assumption: $\rho_t = \rho$ and $\beta = \frac{1}{1+\rho}$

Then

$$u'(c_{t+1}) = u'(c_t) + \epsilon_{t+1} \tag{6.77}$$

MU of consumption is a random walk.

With quadratic utility function, MU $= \alpha - \beta c$, Hall (1978) showed that

$$c_{t+1} = c_t + \epsilon_{t+1}. \tag{6.78}$$

References

[Sargent 1979] TOM SARGENT, "Chapter 9" *Macroeconomic Theory*, 1979