Econ 626: Quantitative Methods II **Fall 2018**

Lecture 5: Calculus of Variations II

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

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5.1 Continue in Continuous Time

Back to the example at beginning of this course (in Section 1.2):

$$\begin{cases} \min & \int_0^T \{c_1(x'(t))^2 + c_2 x(t)\}\mathrm{d}t \\ s.t. & x(0) = 0, x(T) = B, x'(t) \geq 0 \end{cases} \tag{5.1}$$

$$F_x = c_2, \qquad F_{x'} = 2c_1 x'(t), \qquad \frac{\mathrm{d}F_{x'}}{\mathrm{d}t} = 2c_1 x''(t)$$

$$\frac{\mathrm{d}F_{x'}}{\mathrm{d}t} = F_x \Longrightarrow 2c_1 x''(t) = c_2$$

$$x(t) = \frac{c_2}{4c_1}t^2 + k_1 t + k_2$$

$$x(0) = k_2 = 0, \qquad x(T) = \frac{c_2}{4c_1}T^2 + k_1 T = B$$

$\Longrightarrow$

$$k_1 = \frac{B}{T} - \frac{c_2 T}{4c_1} \tag{5.2}$$

$$x^*(t) = \frac{c_2}{4c_1}t^2 + (\frac{B}{T} - \frac{c_2 T}{4c_1})t \tag{5.3}$$

Interpreting the Euler equation

$$c_1(x'(t))^2 = \text{Marginal cost of production} \tag{5.4}$$

$$2c_1 x''(t) = \text{Rate of change in marginal cost} \tag{5.5}$$

$$c_2 = \text{unit inventory holding cost} \tag{5.6}$$

$$\text{Euler Equation}: \quad 2c_1 x''(t) = c_2 \tag{5.7}$$

Another look

$$\int_t^{t+\Delta} 2c_1 x''(s)\mathrm{d}s = \int_t^{t+\Delta} c_2 \mathrm{d}s \tag{5.8}$$

[1] Visit `http://www.luzk.net/misc` for updates.

$$
\begin{aligned}
&2c_1[x'(t+\Delta) - x'(t)] = c_2\Delta \\
\iff &2\underbrace{c_1x'(t) + c_2\Delta}_{(a)} = \underbrace{2c_1x'(t+\Delta)}_{(b)}
\end{aligned} \tag{5.9}
$$

which means that in optimum, we should be indifferent between
(a) producing a unit now (at time t) and storing it for Δ amount of time and
(b) producing a unit at time $t+\Delta$

Brachistchrone Challenge

$$T = \int \mathrm{d}t = \text{Total travel time}$$

$$\mathrm{d}t = \frac{\mathrm{d}s}{\mathrm{d}s/\mathrm{d}t} = \frac{\mathrm{d}s}{v}$$

where ds is a "short" distance travelled.

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$$

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (\underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}}_{y'(x)})^2}$$

$$\mathrm{d}s = \sqrt{1+y'^2}\ \mathrm{d}x$$

Assumption

$$\underbrace{\frac{mv^2}{2}}_{\text{Kinetic energy}} = \underbrace{mgy}_{\text{Potential energy}} \tag{5.10}$$

$$m = \text{mass} \tag{5.11}$$

$$g = \text{gravity acceleration parameter} \tag{5.12}$$

$$mg = \text{weight} \tag{5.13}$$

$$\implies v = \sqrt{2gy} \tag{5.14}$$

$$
\begin{aligned}
\min \int_{x_1}^{x_2} \mathrm{d}t &= \int_{x_1}^{x_2} \frac{\mathrm{d}s}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\mathrm{d}x \\
&= \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \frac{\sqrt{1+y'(x)^2}}{\sqrt{y(x)}}\mathrm{d}x \\
\iff \min &\int_{x_1}^{x_2} \frac{\sqrt{1+y'(x)^2}}{\sqrt{y(x)}}\mathrm{d}x
\end{aligned} \tag{5.15}
$$

FONC (other varsion of the EE, see equation(4.27)):

$$\frac{\mathrm{d}(F - y'F_{y'})}{\mathrm{d}x} = F_t = 0$$

Note that the second equal sign above is not necessarily true in general. This is true in our case.

$$\left[\frac{1+(y'(x))^2}{y(x)}\right]^{1/2} - (y'(x))^2\left[(1+(y'(x))^2)y(x)\right]^{-1/2} = C \tag{5.16}$$

Solution to this differential euqtion is given by

$$x = -(2ky - y^2)^{\frac{1}{2}} + k \arccos(1 - \frac{y}{k}) + \bar{k} \tag{5.17}$$

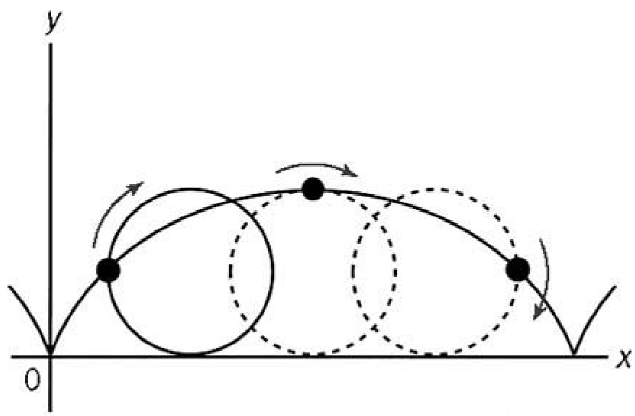

Figure 5.1: the solution is a cycloid (picture from the Internet)

Notes

1. If we have n choice variables, then we need to write EE for each one of them.
2. In case of higher order derivatives, we can proceed either by
   a) introducing new variable, or
   b) using Euler-Poisson Equation.

Example:

$$\int_0^T (ty^2 + yy' + y''^2)\mathrm{d}t \tag{5.18}$$

$$y(0) = A,\ y(T) = B,\ y'(0) = \alpha,\ y'(T) = \beta$$

Let: $z = y' \Longrightarrow z' = y''$

$\Longrightarrow$

$$F = ty^2 + yz + z'^2,\ z(0) = \alpha,\ z(T) = \beta$$

$\Longrightarrow$

$$\begin{cases} \int_0^T ty^2 + yz + z'^2 \mathrm{d}t \\ y(0) = A,\ y(T) = B,\ z(0) = \alpha,\ z(T) = \beta \end{cases} \tag{5.19}$$

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Example: Continuous Time Dynamic Optimization Model

$$\max \quad \int_0^\infty U(C(t))e^{-\delta t}\mathrm{d}t \tag{5.20}$$

$$\text{s.t.} \quad C(t) + B'(t) = \rho(t)B(t) + Y(t) \tag{5.21}$$

where

C = consumption

Y = income

B = Bonds/Saving

$\rho(t)$ = Time-varying interest rate

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General Constained Case

$$\max \quad \int_0^\infty F(y(t), y'(t), t)\mathrm{d}t \tag{5.22}$$

$$\text{s.t.} \quad G(y(t), y'(t), t) = 0, \ \forall t \tag{5.23}$$

The Lagrangian is given by

$$\mathcal{L} = \int_0^\infty F(y(t), y'(t), t) + \lambda(t)G(y(t), y'(t), t)\mathrm{d}t.$$

Euler Equation:

$$\frac{\partial F}{\partial y(t)} + \lambda(t)\frac{\partial G}{\partial y(t)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial F}{\partial y'(t)} + \lambda(t)\frac{\partial G}{\partial y'(t)}\right] = 0 \tag{5.24}$$

EE w.r.t. $C(t)$:

$$U'(C(t))e^{-\delta t} + \lambda(t) = 0 \tag{5.25}$$

$$\implies U'(C(t))e^{-\delta t} = -\lambda(t) \tag{5.26}$$

EE w.r.t. $B(t)$:

$$\lambda(t)(-\rho(t)) - \frac{\mathrm{d}}{\mathrm{d}t}[\lambda(t)] = 0 \tag{5.27}$$

$$\implies -\lambda(t)\rho(t) = \lambda'(t) \tag{5.28}$$

(5.26) $\implies$

$$\ln U'(C(t)) - \delta t = \ln(-\lambda(t))$$

$\implies$

$$\frac{1}{U'(C(t))}U''(C(t))C'(t) - \delta = \frac{-\lambda'(t)}{-\lambda(t)} = -\rho(t)$$

$\implies$

$$-\frac{U''(C(t))}{U'(C(t))}C'(t) = \rho(t) - \delta \tag{5.29}$$

Two equations: (5.21) and (5.29)

Two unknowns: $\{C_t\}_{t=0}^\infty$, $\{B_t\}_{t=0}^\infty$

$$(5.21) \iff B'(t) - \rho(t)B(t) = Y(t) - C(t) \tag{5.30}$$

Recall

$$y'(t) + g(t)y(t) = h(t) \tag{5.31}$$

Forward-looking solution

$$y(t) = Ae^{-\int_0^t g(v)\mathrm{d}v} - e^{-\int_0^t g(v)\mathrm{d}v}\int_t^\infty h(s)e^{\int_t^{t+s} g(v)\mathrm{d}v}\mathrm{d}s \tag{5.32}$$

Claim: $e^{-\int_0^t \rho(s)\mathrm{d}s} = \Phi(t)$ is the discount factor when $\rho = \rho(t)$.

If $\rho(t) = \rho$, $\Phi(t) = e^{-\rho t}$.

$$B(t) = A \underbrace{e^{\int_0^t \rho(v)\mathrm{d}v}}_{[\Phi(t)]^{-1}} - e^{\int_0^t \rho(v)\mathrm{d}v} \int_t^\infty [Y(s) - C(s)] e^{-\int_t^{t+s} \rho(v)\mathrm{d}v} \mathrm{d}s \tag{5.33}$$

Assumption

- $\rho(t) = \rho \quad \Longrightarrow \quad e^{\int_0^t \rho \mathrm{d}v} = e^{\rho t} = [\Phi(t)]^{-1}$
- $Y(t) = Y$ (a constant)

$$B(t) = A[\Phi(t)]^{-1} - [\Phi(t)]^{-1} \int_t^\infty \Phi(s)[Y - C(s)]\mathrm{d}s \tag{5.34}$$

$$B(t) = Ae^{\rho t} - e^{\rho t} \int_t^\infty e^{-\rho s}[Y - C(s)]\mathrm{d}s \tag{5.35}$$

$$e^{\rho t} \int_t^\infty e^{-\rho s} C(s)\mathrm{d}s = B(t) + e^{\rho t} \int_t^\infty e^{-\rho s} Y \mathrm{d}s - Ae^{\rho t} \tag{5.36}$$

If $A > 0$, PDV of C shrinks exponentially.

If $A < 0$, PDV of C grows exponentially.

$\Longrightarrow A = 0$

$$\underbrace{e^{\rho t} \int_t^\infty e^{-\rho s} C(s)\mathrm{d}s = B(t) + e^{\rho t} \int_t^\infty e^{-\rho s} Y \mathrm{d}s}_{\text{life-time budget constraint}} \tag{5.37}$$

Assumption

- CRRA type utility function $U(C) = \ln C$

  (5.29) $\Longrightarrow \left(-\frac{U''}{U'} C\right) \frac{C'}{C} = \rho - \delta$,

  where $-\frac{U''}{U'} C$ is the coefficient of relative risk aversion.

$\Longrightarrow$

$$\frac{C'(t)}{C(t)} = \rho - \delta \tag{5.38}$$

$\Longrightarrow$

$$C(t) = e^{(\rho - \delta)t + a} \tag{5.39}$$

$$C(s) = e^{(\rho - \delta)(s - t)} C(t) \tag{5.40}$$

(5.37) $\Longrightarrow$

$$\begin{aligned}
e^{\rho t}\int_t^{\infty} e^{-\rho s}e^{(\rho-\delta)(s-t)}C(t)\mathrm{d}s &= e^{\rho t}\int_t^{\infty} e^{-\rho s}Y\mathrm{d}s + B(t) && (5.41)\\
C(t)e^{\rho t}\int_t^{\infty} e^{-\rho s}e^{(\rho-\delta)(s-t)}\mathrm{d}s &= e^{\rho t}Y\int_t^{\infty} e^{-\rho s}\mathrm{d}s + B(t) && (5.42)\\
C(t)e^{\rho t}e^{\delta t-\rho t}\int_t^{\infty} e^{-\delta s}\mathrm{d}s &= Ye^{\rho t}\int_t^{\infty} e^{-\rho s}\mathrm{d}s + B(t) && (5.43)\\
C(t)e^{\delta t}\int_t^{\infty} e^{-\delta s}\mathrm{d}s &= Ye^{\rho t}\int_t^{\infty} e^{-\rho s}\mathrm{d}s + B(t) && (5.44)\\
C(t)e^{\delta t}(\frac{e^{-\delta t}}{\delta}) &= Ye^{\rho t}(\frac{e^{-\rho t}}{\rho}) + B(t) && (5.45)\\
\frac{C(t)}{\delta} &= \frac{Y}{\rho} + B(t) && (5.46)\\
C(t) &= \underbrace{\frac{\delta}{\rho}}_{MPC}[Y + \rho B(t)] && (5.47)
\end{aligned}$$

Backward-looking solution (applies when $g(t) > 0$)

$$y(t) = Ae^{-\int_0^t g(v)\mathrm{d}v} + \int_0^{\infty} h(t-s)e^{\int_t^{t-s} g(v)\mathrm{d}v}\mathrm{d}s \tag{5.48}$$

5.2 Discrete time

$P_t = \alpha P_{t-1} + \beta$

$P_0 = \text{given}$

$P_1 = \alpha P_0 + \beta$

$P_2 = \alpha^2 P_0 + \alpha\beta + \beta$

$P_3 = \alpha^3 P_0 + \alpha^2\beta + \alpha\beta + \beta$

$\vdots$

$P_t = \alpha^t P_0 + \sum_{i=0}^{t-1}\alpha^i\beta = \alpha^t P_0 + \frac{\beta}{1-\alpha} - \frac{\beta}{1-\alpha}\alpha^t$

$P_t = \frac{\beta}{1-\alpha} + (P_0 - \frac{\beta}{1-\alpha})\alpha^t$

If $P_0 = \underbrace{\frac{\beta}{1-\alpha}}_{P^*}$, then $P_t = \frac{\beta}{1-\alpha}$ for all t.

`[intert two graphs here for the convergence patterns of` $0 < \alpha < 1$ `and` $-1 < \alpha < 0$`]`

$|\alpha| < 1$ – stable root

$|\alpha| > 1$ – unstable root

Lag Operator (Sargent (1987))

$$\begin{aligned} Lx_t &= x_{t-1} && (5.49)\\ L(Lx_t) &= L^2 x_t = x_{t-2} && (5.50)\\ L^n x_t &= x_{t-n} && (5.51)\\ L^{-1} x_t &= x_{t+1} && (5.52)\\ L^{-n} x_t &= x_{t+n} && (5.53)\\ L^0 &\equiv 1 && (5.54) \end{aligned}$$

Polynomial of Lag Operator

$$A(L) = a_0 + a_1 L + a_2 L^2 + ... = \sum_{j=0}^{\infty} a_j L^j \tag{5.55}$$

$$\begin{aligned} A(L)x_t &= (a_0 + a_1 L + a_2 L^2 + ...)x_t \\ &= a_0 x_t + a_1 x_{t-1} + a_2 x_{t-2} + ... \\ &= \sum_{j=0}^{\infty} a_j x_{t-j} \end{aligned} \tag{5.56}$$

Rational Polynomial

$$A(L) = \frac{B(L)}{C(L)} \tag{5.57}$$

Example:

$$AL = \frac{1}{1 - \lambda L}$$

Claim: If $|\lambda| < 1$, then

$$\frac{1}{1-\lambda L} = 1 + \lambda L + \lambda^2 L^2 + ... = \sum_{i=0}^{\infty} \lambda^i L^i \tag{5.58}$$

If $|\lambda| > 1$, then

$$\frac{1}{1-\lambda L} = 1 + \frac{\lambda L}{1-\lambda L} = 1 - \frac{1}{1-(\frac{1}{\lambda})L^{-1}} = 1 - (1 + \frac{1}{\lambda} L^{-1} + \left(\frac{1}{\lambda}\right)^2 L^{-2} + ...) = -\sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i L^{-i} \tag{5.59}$$

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References