Econ 626: Quantitative Methods II **Fall 2018**

Lecture 4: Calculus of Variations

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

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4.1 The simplest problem of Calculus of Variations

$$\max_{x(t)} \int_{t_1}^{t_2} F(t, x(t), x'(t))\mathrm{d}t \tag{4.1}$$

$$\text{s.t.} \quad x(t_0) = x_0, x(t_1) = x_1 \tag{4.2}$$

Assumptions

1. $F(\cdots)$ is continuous in its three auguments.
2. $F(\cdots)$ has continuous partial derivatives w.r.t. $x(t)$ and $x'(t)$.

Definition 4.1 *A path is feasible or admissible if it is C^1 on $[t_0, t_1]$ and satisfies (4.2).*

FONC (**Euler** Equation)

Suppose that $x^*(t),\ t_0 \leq t \leq t_1$ solves (4.1). Let $x(t)$ be some other feasible path. Let $h(t)$ be:

$$h(t) = x(t) - x^*(t) \tag{4.3}$$

Thus:

- $x(t)$ is a comparision path.
- $h(t)$ is a deviation path.

Since $x(t)$ and $x^*(t)$ both satisfy (4.2), we have

$$h(t_0) = h(t_1) = 0 \tag{4.4}$$

Define $y(t)$ as follows

$$y(t) = x^*(t) + ah(t) \tag{4.5}$$

where a is some parameter.

Then

$$\begin{cases} y(t_0) = x^*(t_0) + ah(t_0) = x^*(t_0) = x_0 \\ y(t_1) = x^*(t_1) + ah(t_1) = x^*(t_1) = x_1 \end{cases} \tag{4.6}$$

It follows that $y(t)$ is a feasible path because it is C^1 for any arbitary a and it satisfies (4.2).

[1] Visit `http://www.luzk.net/misc` for updates.

`[insert a graph of` $[x^*(t), x(t), h(t)]$ `here]`

Hold $x^*(t)$ and $h(t)$ fixed and compute (4.1) for $y(t)$ as a function of a:

$$\begin{aligned} g(a) &\stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, y(t), y'(t))\mathrm{d}t \\ &= \int_{t_0}^{t_1} F(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))\mathrm{d}t \end{aligned} \tag{4.7}$$

By assumption $x^*(t)$ maximizes (4.1). Therefore the function $g(a)$ will attain its maximum where $a = 0$. Therefore, by regular FONC,

$$g'(a)\Big|_{a=0} = g'(0) = 0 \tag{4.8}$$

Leibnitz Theorem

Let $f(x, r)$ be continuous w.r.t. x, $\forall r$, and let $\frac{\partial f(x,r)}{\partial r}$ be continuous in the rectangle $a \le x \le b$, $\underline{r} \le r \le \bar{r}$ in the x-r plane. Let the functions $A(r)$ and $B(r)$ be C^1. If $V(r) = \int_{A(r)}^{B(r)} f(x, r)\mathrm{d}x$, then

$$V'(r) = f(B(r), r)B'(r) - f(A(r), r)A'(r) + \int_{A(r)}^{B(r)} \frac{\partial f(x, r)}{r}\mathrm{d}x \tag{4.9}$$

By Leibnitz Theorem

$$\begin{aligned} g'(a) &= F(t, x^*(t_1(a)) + ah(t_1(a)), x^{*\prime}(t_1(a)) + ah'(t_1(a)))(\cancelto{0}{\frac{\mathrm{d}t_1}{\mathrm{d}a}}) \\ &\quad -F(t, x^*(t_0(a)) + ah(t_0(a)), x^{*\prime}(t_0(a)) + ah'(t_0(a)))(\cancelto{0}{\frac{\mathrm{d}t_0}{\mathrm{d}a}}) \\ &\quad + \int_{t_0}^{t_1} \Big[F_x(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))h(t) \\ &\quad +F_{x'}(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))h'(t)\Big]\mathrm{d}t \\ &= \int_{t_0}^{t_1} \Big[F_x(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))h(t) + F_{x'}(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))h'(t)\Big]\mathrm{d}t \end{aligned} \tag{4.10}$$

$$g'(0) = \int_{t_0}^{t_1} \Big[F_x(t, x^*(t), x^{*\prime}(t))h(t) + F_{x'}(t, x^*(t), x^{*\prime}(t))h'(t)\Big]\mathrm{d}t = 0 \quad \text{by FONC} \tag{4.11}$$

which is a necessary condition for a maximum.

Note: (4.11) must hold for all h that is continuous and satisfies (4.4).

Recall: Integration by part

$$\int_{t_0}^{t_1} \Big[F_{x'}(t, x^*(t), x^{*\prime}(t))h'(t)\Big]\mathrm{d}t = [F_{x'}(t, x^*(t), x^{*\prime}(t))h(t)]\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \Big[h(t)\frac{\mathrm{d}}{\mathrm{d}t}F_{x'}(t, x^*(t), x^{*\prime}(t))\Big]\mathrm{d}t \tag{4.12}$$

$$= 0 - \int_{t_0}^{t_1} \Big[h(t)\frac{\mathrm{d}}{\mathrm{d}t}F_{x'}(t, x^*(t), x^{*\prime}(t))\Big]\mathrm{d}t \tag{4.13}$$

$$\begin{aligned} g'(0) &= \int_{t_0}^{t_1} \Big[F_x(t, x^*(t), x^{*\prime}(t))h(t) + F_{x'}(t, x^*(t), x^{*\prime}(t))h'(t) \Big] \mathrm{d}t && (4.14) \\ &= \int_{t_0}^{t_1} \Big[F_x(t, x^*(t), x^{*\prime}(t))h(t) \Big] \mathrm{d}t + \int_{t_0}^{t_1} \Big[F_{x'}(t, x^*(t), x^{*\prime}(t))h'(t) \Big] \mathrm{d}t && (4.15) \\ &= \int_{t_0}^{t_1} \Big[F_x(t, x^*(t), x^{*\prime}(t))h(t) \Big] \mathrm{d}t - \int_{t_0}^{t_1} \Big[h(t) \frac{\mathrm{d}}{\mathrm{d}t} F_{x'}(t, x^*(t), x^{*\prime}(t)) \Big] \mathrm{d}t && (4.16) \\ &= \int_{t_0}^{t_1} \Big[F_x(t, x^*(t), x^{*\prime}(t)) - \frac{\mathrm{d}}{\mathrm{d}t} F_{x'}(t, x^*(t), x^{*\prime}(t)) \Big] h(t) \mathrm{d}t = 0 && (4.17) \end{aligned}$$

(4.17) must hold true if $x^*(t)$ maximizing (4.1). Moreover, it must hold for all $h(t)$ that is C^1 and satisfies (4.4). This will be true only if the term in the brackets vanishes:

$$F_x - \frac{\mathrm{d}}{\mathrm{d}t} F_{x'} = 0, \; t_0 \le t \le t_1 \tag{4.18}$$

which is a necessary condition for a maximum. (4.18) is called Euler Equation. The Euler equation holds true because of the following lemma.

Theorem 4.2 (The Fundamental Lemma of Calculus of Variations) *Suppose that $g(t)$ is a continuous function defined on $[t_0, t_1]$. If*

$$\int_{t_0}^{t_1} g(t)h(t)\mathrm{d}t = 0 \tag{4.19}$$

for all continuous $h(t)$ defined on $[t_0, t_1]$ and satisfies (4.4), then $g(t) = 0 \; \forall t \in [t_0, t_1]$.

Proof: Suppose that $g(t) \neq 0$ for some $\bar{t} \in [t_0, t_1]$. WLOG, assume $g(\bar{t}) = m > 0$. Since $g(t)$ is continuous at $t = \bar{t}$, for $\epsilon = \frac{m}{2}$, we know there exist $\delta_\epsilon > 0$ such that,

$$g(t) \ge g(\bar{t}) - \epsilon = m - \frac{m}{2} = \frac{m}{2}, \qquad \forall t \in (\bar{t} - \delta_\epsilon, \bar{t} + \delta_\epsilon) \cap [t_0, t_1] \tag{4.20}$$

Let $h(t)$ be

$$h(t) = \begin{cases} (t-a)(b-t), & \text{if } t \in [a,b] \\ 0, & \text{elsewhere} \end{cases} \tag{4.21}$$

with $a = \max\{\bar{t} - \delta_\epsilon, t_0\}$, $b = \min\{\bar{t} + \delta_\epsilon, t_1\}$, and $b - a > 0$ for δ_ϵ sufficiently small. (Also note that $h(t_0) = h(t_1) = 0$ and $h(\cdot)$ is continuous.)

$$\int_{t_0}^{t_1} g(t)h(t)\mathrm{d}t = \int_a^b g(t)(t-a)(b-t)\mathrm{d}t \ge \int_a^b \frac{m}{2}(t-a)(b-t)\mathrm{d}t = \frac{(b-a)^3}{6} > 0 \tag{4.22}$$

which contradicts the hypothesis that (4.19) holds for all $h(t)$ with the required properties. ■

Other versions of the Euler equation

$$\frac{\mathrm{d}F_{x'}}{\mathrm{d}t} = F_{x't} + F_{x'x}x' + F_{x'x'}x'' \tag{4.23}$$

$$\Longrightarrow F_x = F_{x't} + F_{x'x}x' + F_{x'x'}x'' \tag{4.24}$$

where the partial derivatives must be evaluated at $(t, x^*, x^{*\prime})$ and $x' = x'(t)$, $x'' = x''(t)$. Note also that (4.24) is a second order differential equation.

Integrate (4.18) $\Longrightarrow$

$$F_{x'} = \int F_x + C_1 \tag{4.25}$$

which is known as du Bois-Reymond equation.

Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}[F - x'F_{x'}] = F_t + F_x x' + \cancel{F_{x'}x''} - x'\frac{\mathrm{d}F_{x'}}{\mathrm{d}t} - \cancel{x''F_{x'}} = F_t + x'\overbrace{\cancel{(F_x - \frac{\mathrm{d}F_{x'}}{\mathrm{d}t})}}^{0 \text{ by EE}} \tag{4.26}$$

$\Longrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}[F - x'F_{x'}] = F_t \tag{4.27}$$

which is particularly useful if F does not depend on t directly, i.e. $F_t = 0$.

Example: $F(\cdot) = (x'(t))^2$

$$\begin{cases} \min & \int_0^T (x'(t))^2 \mathrm{d}t \\ \text{s.t.} & x(0) = 0,\ x(T) = B \end{cases} \tag{4.28}$$

■

$$F_{x'} = 2x'(t), F_x = 0 \tag{4.29}$$

EE $\Longrightarrow$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}[2x'(t)] \tag{4.30}$$

$$\Longrightarrow x''(t) = 0 \tag{4.31}$$

$$x(t) = c_1 t + c_2 \tag{4.32}$$

With $x(0) = 0,\ x(T) = B$, we get

$$x(t) = \frac{B}{T}t. \tag{4.33}$$

References