Econ 626: Quantitative Methods II **Fall 2018**

Lecture 3: Dynamic Programming III

Lecturer: Prof. Daniel Levy *Scribes: Zhikun Lu*

Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

Date: August 31, 2018

3.1 Value function iteration

Inital guess: $V_0(K_{T+1}) = 0$

$$\Longrightarrow V_1(K_T) = \begin{cases} \max_{\{C_T, K_{T+1}\}} [u(C_T) + \beta V_0(K_{T+1})] \\ s.t. \quad C_T + K_{T+1} = K_T^\alpha \end{cases} \tag{3.1}$$

Recall that $K_{T+1} = 0$ because it is the last period.

$$\Longrightarrow \begin{cases} K_{T+1} = 0 \\ C_T = K_T^\alpha \end{cases} \tag{3.2}$$

Plugging (3.2) into (3.1), we get

$$V_1(K_T) = \ln(K_T^\alpha). \tag{3.3}$$

Let's continue:

$$V_2(K_{T-1}) = \begin{cases} \max_{\{C_{T-1}, K_T\}} [u(C_{T-1}) + \beta V_1(K_T)] \\ s.t. \quad C_{T-1} + K_T = K_{T-1}^\alpha \end{cases} \tag{3.4}$$

$$\Longrightarrow V_2(K_{T-1}) = \begin{cases} \max_{\{C_{T-1}, K_T\}} u(C_{T-1}) + \beta \ln(K_T^\alpha) \\ s.t. \quad C_{T-1} + K_T = K_{T-1}^\alpha \end{cases} \tag{3.5}$$

$$\mathcal{L} = \ln C_{T-1} + \beta \ln(K_T^\alpha) + \lambda[K_{T-1}^\alpha - C_{T-1} - K_T] \tag{3.6}$$

FONC

$$\frac{1}{C_{T-1}} - \lambda = 0 \tag{3.7}$$

$$\beta(\frac{1}{K_t^\alpha})(\alpha K_T^{\alpha-1}) - \lambda = 0 \tag{3.8}$$

$$\Longrightarrow \lambda = \frac{\alpha\beta}{K_T} \tag{3.9}$$

$$\lambda = \frac{1}{C_{T-1}} \tag{3.10}$$

$$\frac{\alpha\beta}{K_T} = \frac{1}{C_{T-1}} \quad \text{or} \quad C_{T-1} = \frac{K_T}{\alpha\beta} \tag{3.11}$$

[1]Visit `http://www.luzk.net/misc` for updates.

<u>Plug it into the constraint</u>

$$\frac{K_T}{\alpha\beta} + K_T = K_{T-1}^{\alpha} \tag{3.12}$$

$$\Longrightarrow K_T = \frac{\alpha\beta}{1+\alpha\beta} K_{T-1}^{\alpha} \tag{3.13}$$

$$C_{T-1} = \frac{K_T}{\alpha\beta} = \frac{1}{\cancel{\alpha\beta}} \frac{\cancel{\alpha\beta}}{1+\alpha\beta} K_{T-1}^{\alpha} = \frac{1}{1+\alpha\beta} K_{T-1}^{\alpha} \tag{3.14}$$

Plug (3.12) and (3.14) into (3.5)

$$\begin{aligned} V_2(K_{T-1}) &= \max_{\{C_{T-1}, K_T\}} u(C_{T-1}) + \beta \ln(K_T^{\alpha}) \quad \text{s.t. ...} &(3.15)\\ &= \ln(\frac{1}{1+\alpha\beta} K_{T-1}^{\alpha}) + \beta \ln[\frac{\alpha\beta}{1+\alpha\beta} K_{T-1}^{\alpha}]^{\alpha} &(3.16)\\ &= \alpha\beta \ln \alpha\beta - (1+\alpha\beta)\ln(1+\alpha\beta) + (1+\alpha\beta)\ln K_{T-1}^{\alpha} &(3.17) \end{aligned}$$

$$V_3(K_{T-2}) = \begin{cases} \max\limits_{\{C_{T-2}, K_{T-1}\}} [u(C_{T-2}) + \beta V_2(K_{T-1})] \\ s.t. \quad C_{T-2} + K_{T-1} = K_{T-2}^{\alpha} \end{cases} \tag{3.18}$$

$$\vdots \tag{3.19}$$

It turns out that this sequence of value functions converges to:

$$V(K_t) = \frac{\beta}{1-\beta}[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta] + \frac{\alpha}{1-\alpha\beta}\ln K_t \tag{3.20}$$

To check if this limit function is indeed a solution, we plug it into the the Bellman equation (of the infinite horizon model):

$$V(K_t) = \max \left\{ \ln C_t + \frac{\beta}{1-\beta}[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta] + \frac{\alpha\beta}{1-\alpha\beta}\ln K_{t+1} \right\} \quad s.t. \quad C_t + K_{t+1} = K_t^{\alpha} \tag{3.21}$$

Recall

$$\frac{1}{C_t} = \beta V'(K_{t+1}) \Longrightarrow \frac{1}{\beta C_t} = V'(K_{t+1}) = \frac{\alpha}{1-\alpha\beta}\frac{1}{K_{t+1}} \quad \text{(by taking the derivative of (3.20))} \tag{3.22}$$

Hence

$$\frac{C_t}{K_{t+1}} = \frac{1-\alpha\beta}{\alpha\beta} \tag{3.23}$$

Using the constaint $C_t + K_{t+1} = K_t^{\alpha}$, we can get

$$\frac{K_t^{\alpha} - K_{t+1}}{K_{t+1}} = \frac{1-\alpha\beta}{\alpha\beta} \tag{3.24}$$

$$\Longrightarrow \text{saving rate} = \frac{K_{t+1}}{K_t^{\alpha}} = \alpha\beta \tag{3.25}$$

same as old.

References