Econ 626: Quantitative Methods II **Fall 2018**

Lecture 1: August 27-29

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Disclaimer: *Zhikun is fully responsible for the errors and typos appeared in the notes.*

1.1 Some history

1.1.1 About the Bernoulli brothers

1.1.2 Brachistochrone problem

Brachistochrone means *shortest time*. Note that the shortest path is not the fastest one.

1.2 Methods of solving dynamic optimization

1. Calculus of variations (Newton, ...), 1696-1697
2. Optimal control (Pontryign, ...), 1960-1961
3. Dynamic programming (Bellman), 1957

Example: Suupose that a firm receives an order for B units of a product to be delivered by time T. Its goal is to accomplish this at minimum cost.

`Assumptions`: Unit production cost rises linearly with the production rate, given by

$$c_1 x'(t), c_1 > 0$$

- $x(t)$ - inventory accumulated by time t
- $x'(t)$ - production rate
- $c_2 > 0$ - unit cost of holding inventory per unit of time

Total cost:

$$c_1 x'(t) x'(t) + c_2 x(t) = c_1 (x'(t))^2 + c_2 x(t)$$

This is a continuous time problem:

$$\min \int_0^T [c_1 (x'(t))^2 + c_2 x(t)] \mathrm{d}t$$

$$\text{s.t. } x(0) = 0, x(T) = B, x'(t) \geq 0$$

[1] This note contains 3-days lectures. Visit `http://www.luzk.net/misc` for updates.

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Goal: Find $x(t)$ which will minimize the total cost of production.

Guess: One possible solution is to produce at a constant rate by setting $x'(t) = \frac{B}{T}$. This is indeed feasible:

$$x(t) = \int_0^t \frac{B}{T}\mathrm{d}t = \frac{B}{T}t \tag{1.1}$$

because $x(0) = 0$, $x(T) = B$, and $x'(t) \geq \frac{B}{T}$. In that case,

$$\int_0^T [c_1(x'(t))^2 + c_2 x(t)]\mathrm{d}t = \int_0^T \left[c_1(\frac{B}{T})^2 + c_2\frac{B}{T}t\right]\mathrm{d}t = \left[c_1(\frac{B}{T})^2 t + c_2\frac{B}{2T}t^2\right]\Bigg|_0^T = c_1\frac{B^2}{T} + c_2\frac{BT}{2} \tag{1.2}$$

A Simplified Version: Suppose that $c_2 = 0, c_1 = 1$. The problem is still dynamic in nature. Now the problem becomes

$$\min \int_0^T (x'(t))^2\mathrm{d}t$$
$$\text{s.t. } x(0) = 0, x(T) = B, x'(t) \geq 0 \tag{1.3}$$

Discrete Approximation: Let us convert (1.3) into discrete time setting. Divide the interval $[0, T] \in \mathbb{R}$ into $\frac{T}{k}$ segments with equal length of k.

`[Insert a graph here]`

- Approximation of $x(t)$

  $x(t)$ can be approximated by polygonal line with verticies y at the end point of each segment:

  $$(0, 0), (k, y_1), (2k, y_2), ..., (T, B)$$

  .

`[Insert a graph here]`

- Approximation of $x'(t)$

  $$x'(t) \approx \frac{\Delta x}{\Delta t} = \frac{y_i - y_{i-1}}{k}.$$

Then the objective of the firm is to determine $y_i, i = 1, ..., \frac{T}{k} - 1$ (since the last period value $y_{\frac{T}{k}} = B$), which minimizes the following

$$\min \sum_{i=1}^{\frac{T}{k}} \left[\frac{y_i - y_{i-1}}{k}\right]^2 k$$
$$y_0 = 0, y_{\frac{T}{k}} = B, y_i - y_{i-1} \geq 0 \tag{1.4}$$

(1.4) is the discrete analogue of (1.3).

- FONC w.r.t. y_i:

$$2\frac{y_i - y_{i-1}}{k} - 2\frac{y_{i+1} - y_i}{k} = 0 \iff y_i - y_{i-1} = y_{i+1} - y_i \text{ or } \Delta y_i = \Delta y_{i+1} \tag{1.5}$$

- The solution property:

  Each 'day' the firm produces the same amount $\longrightarrow$ optimal strategy is to produce at `the constant rate`. The analogue between (1.3) and (1.4) suggests that constant production rate might be optimal in the continuous case as well.

`[Insert a graph here]`

Claim 1.1 *Constant production rate is optimal for the continuous time version, (1.3).*

Remark 1.2 *This should not be surprising since "*$(1.3) = \lim_{k\to 0}(1.4)$*".*

Proof: Let $z(t)$ be some other C^1 feasible path $\Longrightarrow z(0) = 0$ and $z(T) = B$.

Define $h(t) \equiv z(t) - x(t)$. Here, $h(t)$ is called a deviation path, $z(t)$ is called a comparison path. Since both $x(t), z(t)$ are feasible, we have

$$h(0) = h(T) = 0$$

$$z(t) = h(t) + x(t)$$

$$z'(t) = h'(t) + x'(t) = h'(t) + \frac{B}{T}$$

$$\begin{aligned}
\int_0^T (z'(t))^2 \mathrm{d}t - \int_0^T (x'(t))^2 \mathrm{d}t &= \int_0^T \left(h'(t) + \frac{B}{T}\right)^2 \mathrm{d}t - \int_0^T \left(\frac{B}{T}\right)^2 \mathrm{d}t \\
&= \int_0^T \left[(h'(t))^2 + 2h'(t)\frac{B}{T} + (\frac{B}{T})^2\right] \mathrm{d}t - \int_0^T \left(\frac{B}{T}\right)^2 \mathrm{d}t \\
&= \int_0^T (h'(t))^2 \mathrm{d}t + \int_0^T 2h'(t)\frac{B}{T}\mathrm{d}t \\
&= \int_0^T (h'(t))^2 \mathrm{d}t + \left[2h(t)\frac{B}{T}\right]\Big|_0^T \\
&= \int_0^T (h'(t))^2 \mathrm{d}t + 2\frac{B}{T}(h(T) - h(0)) \\
&= \int_0^T (h'(t))^2 \geq 0
\end{aligned}$$

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1.3 Dynamic Optimization Framework

In general, a path can be identified if we know

1. starting time $\mathbf{t_0}$
2. starting state $\mathbf{x(t_0)}$
3. direction of path $\mathbf{x'(t)}$

In general, the simplest general calculus of variations problem:

$$\int_{t_0}^{t_1} F(t, x(t), x'(t))\mathrm{d}t$$
$$\text{s.t. } x(t_0) = x_0, x(t_1) = x_1 \tag{1.6}$$

- x - choice vaiable, can be a vector
- we can have higher order derivatives

Note: FONCs of a continuous (discrete) time dynamic optimization model is a differential (difference) equations of several orders.

August 28, 2018 Continued

Objective Function

Value of the variable of interest - utility or profits, addded up over time. Examples include

$$\text{Discrete - } \sum_{t=0}^{\infty} u(c_t), \sum_{t=0}^{\infty} \Pi(P_t)$$
$$\text{Continuous - } \int_0^{\infty} u(c(t))\mathrm{d}t, \int_0^{\infty} \Pi(P(t))\mathrm{d}t$$

$$\sum_{t=0}^{\infty} u(c_t) = u(c_0) + u(c_1) + ...$$

Discounting in Discrete Time

- $\beta =$ `discount rate`, $0 < \beta < 1$
- $\beta^t =$ `discount factor`
- Life-time utility $= \sum_{t=0}^{\infty} \beta^t u(c_t)$
  Also called `Present Discountied Value of the lifetime utility (PDV)`

Discounting in Continuous Time

If we invest \$P at interest rate r/year, then

- after one year, $P + rP = (1 + r)P$
- after two years, $(1 + r)^2 P$
- after t years, $(1 + r)^t P$

If interest is paid twice a year, then

- after 6 months, $P + \frac{r}{2}P = (1 + \frac{r}{2})P$
- after 1 year, $(1 + \frac{r}{2})^2 P$

- after t years, $(1+\frac{r}{2})^{2t}P$

Generally, if interest is paid m times per year, where the per period rate is $\frac{r}{m}$, then

- after 1 year, $(1+\frac{r}{m})^{m}P$
- after t years, $(1+\frac{r}{m})^{mt}P$

In the limit, if we discount continuously, $m \longrightarrow \infty$:

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt} = e^{rt} \tag{1.7}$$

If we invest \$$P$ today at interest rate r, computed continuously, then the amount will grow to \$$Pe^{rt}$. Conversely, `today's value of time t` \$$P$ should be \$$Pe^{-rt}$. Hence, $e^{-rt} \equiv$ continuous time discount factor.

$$\Longrightarrow \max \int_0^\infty u(c(t))e^{-\delta t}\mathrm{d}t,\ t \in \mathbb{R}$$

<u>Comments</u>

- $0 < \delta < 1$ is the discount rate.
- $e^{-\delta t}$ is the discount factor.
- As $t\uparrow$, we have $e^{-\delta t}\downarrow$.
- Generally, $\delta = \delta(t)$. Uzawa (1961)

Depreciation (Decay)

Say, the stock of capital depreciates at rate $b > 0$

$$\Longrightarrow \frac{K'(t)}{K(t)} = -b$$

$$K'(t) + bK(t) = 0$$

First order linear differential equation

$$e^{bt}[K'(t) + bK(t)] = 0$$

$$\int e^{bt}[K'(t) + bK(t)]\mathrm{d}t = C_1$$

$$e^{bt}K(t) + C_2 = C_1$$

$$K(t) = C_3 e^{-bt}$$

$\Longrightarrow$ b is the exponential rate of depreciation. If $K(0) = C_3$ is known, then $K(t) = K(0)e^{-bt}$

Note

If $K(100)$ is known, say $K(100) = C_3 e^{-100b}$, then

$$K(t) = K(100)e^{(100-t)b}$$

`In discrete time`
$K_{t+1} = (1-\delta)K_t$, or $K_{t+1} - (1-\delta)K_t = 0$, which ia s first order linear difference equation.

1.3.1 Dynamic Models

- Infinite horizon v.s. finite horizon
- Discrete time v.s. continuous time
- Deterministic v.s. Stochastic
- Linear v.s. nonlinear

1.4 Dynamic Programming

Consider the following dynamic discret-time, infinite horizon, deterministic model:

$$\max \sum_{t=0}^{\infty} \beta^t u(C_t), \quad 0 < \beta < 1 \tag{1.8}$$

$$\text{s.t.} \quad C_t + I_t = f(K_t) \tag{1.9}$$

- K_t - accumulated by the end of period t-1
- Capital evolution equation:
$$\begin{aligned} K_{t+1} =& I_t + (1-\delta)K_t \\ =& K_t - \delta K_t + I_t \end{aligned} \tag{1.10}$$
- Assumption:
  1. $\delta = 1$
  2. disposible equipment

$$I_t = K_{t+1} \qquad \text{(think about saving)} \tag{1.11}$$

Then (1.9) becomes

$$C_t + K_{t+1} = f(K_t) \tag{1.12}$$

(1.8) & (1.9) $\implies$

$$\max \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{1.13}$$

$$\text{s.t.} \quad C_t + K_{t+1} = f(K_t) \tag{1.14}$$

`Choice variable`: $\{C_t\}_{t=0}^{\infty}$, and $\{K_{t+1}\}_{t=0}^{\infty}$ with K_0 given.

This is a dynamic programming problem. It's solutions are called `policy functions` because the solutions will offer rules about how to choose the optimal values of choice variables as functions of the state variables.

In our case, a policy function will be a rule that will tell the decision maker how to choose optimally C_t and K_{t+1}, give K_t.

1.4.1 Finite Horizon Version

Consider a finite-horizon version of (1.13)-(1.14):

$$\max \quad \sum_{t=0}^{T} \beta^t u(C_t) \tag{1.15}$$

$$\text{s.t.} \quad C_t + K_{t+1} = f(K_t) \tag{1.16}$$

where we choose $\{C_t, K_{t+1}\}_{t=0}^{T}$.

Notice the recursive structure of the dynamic programming problem: Each period's decision problem is identical to other periods' decision problems, thus we can treat a given dynamic programming problem as a sequence of static problems.

`Bellman's insight`
`[Insert a graph here]`

Plug (1.16) in to (1.15)

$$\max_{\{K_t\}_{t=1}^{T}} \quad \sum_{t=0}^{T} \beta^t u(f(K_t) - K_{t+1}) \tag{1.17}$$

FONC w.r.t. K_{t+1}:

$$-\beta^t u'(f(K_t) - K_{t+1}) + \beta^{t+1} u'(f(K_{t+1}) - K_{t+2}) f'(K_{t+1}) = 0 \tag{1.18}$$

which applies to $t < T$, since K_{T+1} is useless. Rewrite the FONC

$$u'(f(K_t) - K_{t+1}) = \beta u'(f(K_{t+1}) - K_{t+2}) f'(K_{t+1}) \tag{1.19}$$

which is a `second-order difference equation`.

Using $C_t = f(K_t) - K_{t+1}$, we can get `the Euler equation`

$$u'(C_t) = \beta u'(C_{t+1}) f'(K_{t+1}). \tag{1.20}$$

`Two Period Model`
`[Insert a graph here]`

The Last Period

When t = T, the last period, we will want to consume everything because there is no tomorrow.

$$K_{T+1} = 0 \Longrightarrow C_{T+1} = f(K_T)$$

Thus our problem becomes

$$\max \quad \sum_{t=0}^{T} \beta^t u(C_t) \tag{1.21}$$

$$\begin{aligned} \text{s.t.} \quad C_t + K_{t+1} &= f(K_t) \\ K_0 \text{ given and } K_{T+1} &= 0 \end{aligned} \tag{1.22}$$

Example: Let $u(C) = \ln C$, $f(K) = K^\alpha$, $0 < \alpha < 1$, then the Euler equation becomes

$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}} \alpha K_{t+1}^{\alpha-1} \tag{1.23}$$

$$\text{or} \qquad \frac{1}{K_t^\alpha - K_{t+1}} = \frac{\beta}{K_{t+1}^\alpha - K_{t+2}} \alpha K_{t+1}^{\alpha-1}, \tag{1.24}$$

which is (1.19) with specific utility and production functions.

$$\begin{aligned} \frac{1}{K_t^\alpha - K_{t+1}} &= \frac{\beta}{K_{t+1}^\alpha - K_{t+2}} \alpha K_{t+1}^{\alpha-1} \\ \Longleftrightarrow \frac{\frac{1}{K_t^\alpha}}{\frac{1}{K_t^\alpha}(K_t^\alpha - K_{t+1})} &= \frac{\alpha\beta \frac{1}{K_{t+1}^\alpha} K_{t+1}^{\alpha-1}}{\frac{1}{K_{t+1}^\alpha}(K_{t+1}^\alpha - K_{t+2})} \\ \Longleftrightarrow \frac{1}{K_t^\alpha} \frac{1}{1 - \frac{K_{t+1}}{K_t^\alpha}} &= \frac{\beta}{1 - \frac{K_{t+2}}{K_{t+1}^\alpha}} \alpha \frac{K_{t+1}^{\alpha-1}}{K_{t+1}^\alpha} \end{aligned} \tag{1.25}$$

Define $w_t \equiv \frac{K_{t+1}}{K_t^\alpha}$ = saving rate, then

$$\Longrightarrow \frac{w_t}{1 - w_t} = \frac{\alpha\beta}{1 - w_{t+1}} \tag{1.26}$$

We have transformed the original 2nd order difference equation into a 1st order difference equation system

$$\begin{cases} \frac{w_t}{1 - w_t} = \frac{\alpha\beta}{1 - w_{t+1}} \\ w_t = \frac{K_{t+1}}{K_t^\alpha} \end{cases} \tag{1.27}$$

(1.26) can be rewritten as

$$w_{t+1} = 1 + \alpha\beta - \alpha\beta \frac{1}{w_t} \tag{1.28}$$

Plot it with $\alpha = 0.8, \beta = 0.8$:

$$w_{t+1} = 1.6 - \frac{0.64}{w_t} \tag{1.29}$$

Let $w_{t+1} = w_t = w^*$, solving $w = 1 + \alpha\beta - \frac{\alpha\beta}{w}$ yields

$$w_{1,2} = \begin{cases} \frac{(1 + \alpha\beta) + \sqrt{(1 + \alpha\beta)^2 - 4\alpha\beta}}{2} = 1 \\ \frac{(1 + \alpha\beta) - \sqrt{(1 + \alpha\beta)^2 - 4\alpha\beta}}{2} = \alpha\beta \end{cases} \tag{1.30}$$

However, w cannot be 1. It is not optimal because it means that consumption equals zero, which does not make sense. We rule it out. Hence,

$$w^* = \alpha\beta \tag{1.31}$$

To proceed, let us compute w's a follows:

$$w_t = \frac{K_{t+1}}{K_t} \Longrightarrow \frac{K_{T+1}}{K_T} = 0 \tag{1.32}$$

since $K_{T+1} = 0$.

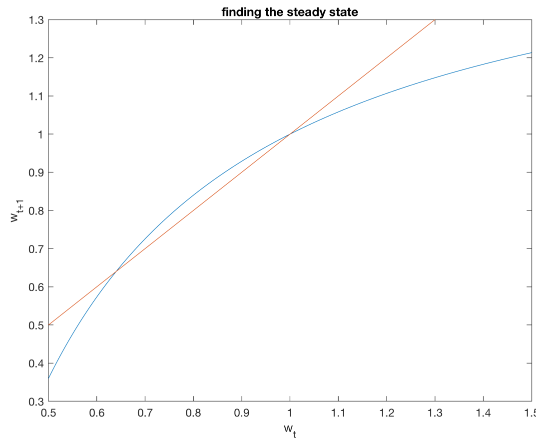

Figure 1.1: The red line is the 45-degree line and the two intersections are possible steady states.

As we are working backwards, rewrite (1.26) as

$$w_t = \frac{\alpha\beta}{1+\alpha\beta - w_{t+1}}. \tag{1.33}$$

Hence

$$w_{T-1} = \frac{\alpha\beta}{1+\alpha\beta-w_T} = \frac{\alpha\beta}{1+\alpha\beta} \tag{1.34}$$

$$w_{T-2} = \frac{\alpha\beta}{1+\alpha\beta-w_{T-1}} = \frac{\alpha\beta}{1+\alpha\beta-\frac{\alpha\beta}{1+\alpha\beta}} = \frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} \tag{1.35}$$

$$w_{T-3} = \frac{\alpha\beta}{1+\alpha\beta-w_{T-2}} = \frac{\alpha\beta}{1+\alpha\beta-\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}} = \frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3} \tag{1.36}$$

(by guess and verify)

$$w_{T-4} = \frac{\alpha\beta}{1+\alpha\beta-w_{T-3}} = \frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3+(\alpha\beta)^4}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3+(\alpha\beta)^4} \tag{1.37}$$

$$\vdots$$

$$w_{T-t} = \frac{\sum_{s=0}^{t}(\alpha\beta)^s - 1}{\sum_{s=0}^{t}(\alpha\beta)^s} \qquad (t = 1, 2, \ldots, T-1) \tag{1.38}$$

$$\vdots$$

$$w_1 = w_{T-(T-1)} = \frac{\sum_{s=0}^{T-1}(\alpha\beta)^s - 1}{\sum_{s=0}^{T-1}(\alpha\beta)^s} \tag{1.39}$$

$$w_0 = w_{T-T} = \frac{\sum_{s=0}^{T}(\alpha\beta)^s - 1}{\sum_{s=0}^{T}(\alpha\beta)^s} \tag{1.40}$$

By re-labelling, we can get the general formula for $0 \le t \le T-1$:

$$w_t = \frac{\sum_{s=0}^{T-t}(\alpha\beta)^s - 1}{\sum_{s=0}^{T-t}(\alpha\beta)^s} \tag{1.41}$$

Recall $\sum_{s=0}^{n} b^s = \frac{1-b^{n+1}}{1-b}$, and we can rewrite (1.41):

$$w_t = \frac{\sum_{s=0}^{T-t}(\alpha\beta)^s - 1}{\sum_{s=0}^{T-t}(\alpha\beta)^s} = w_t = \frac{\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)} - 1}{\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)}} = \frac{\alpha\beta - (\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+1}}, \quad t = 0, 1, \ldots, T-1. \tag{1.42}$$

Recall $w_t = \frac{K_{t+1}}{K_t^\alpha}$, we have

$$K_{T+1} = w_T K_t^\alpha = \frac{\alpha\beta - (\alpha\beta)^{T-T+1}}{1-(\alpha\beta)^{T-T+1}} K_t^\alpha = \frac{\alpha\beta - (\alpha\beta)^1}{1-(\alpha\beta)^1} K_t^\alpha = 0 \tag{1.43}$$

which is consistent with $K_{T+1} = 0$.

Now let $T \longrightarrow \infty$,

$$\lim w_t = \alpha\beta = w^*$$

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1.4.2 Infinite Horizon Model

Back to the infinite horizon model

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{1.44}$$

$$\begin{aligned} \text{s.t.} \quad C_t + K_{t+1} &= f(K_t) \\ K_0 &\text{ given} \end{aligned} \tag{1.45}$$

Note: Since this is an infinite horizon problem, we cannot proceed with backward recursion.

Instead, we will use forward recursion. For this, we require additive-separability:

$$\sum_{t=s}^{\infty} u(C_t) = u(C_s) + \sum_{t=s+1}^{\infty} u(C_t) \tag{1.46}$$

which implies time-separability – $MU(C_t) = MU(C_{t+1})$.

As we saw,

$$K_{t+1} = g(K_t), \quad t = 0, 1, \ldots \tag{1.47}$$

where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the saving function, which is also called `policy function` in the language of Dynamic Programming.

`Value function` is the value of the objective function in optimum, i.e., the maximized value of the objective function.

By (1.47), we have

$$K_1 = g(K_0) \tag{1.48}$$

$$K_2 = g(K_1) = g(g(K_0)) \equiv \bar{g}(K_0) \tag{1.49}$$

$$K_3 = g(K_2) = g(\bar{g}(K_0)) \equiv \bar{\bar{g}}(K_0) \tag{1.50}$$

$$\vdots$$

Hence the whole sequence of choice variables, $\{C_t, K_{t+1}\}_{t=0}^{\infty}$, depends on K_0, so does the the value function $V(\cdot)$:

$$V(K_0) = \begin{cases} \max\limits_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(C_t) \\ \text{s.t.} & C_t + K_{t+1} = f(K_t), \quad K_0 \text{ given} \end{cases} \tag{1.51}$$

Let us simplify the problem by breaking it into two problems.

$$V(K_0) = \begin{cases} \max\limits_{\{C_0,K_1\}} \left[u(C_0) + \begin{cases} \max\limits_{\{C_t,K_{t+1}\}_{t=1}^{\infty}} & \sum_{t=1}^{\infty} \beta^t u(C_t) \\ \text{s.t.} & C_t + K_{t+1} = f(K_t), \quad K_1 \text{ given} \end{cases} \right] \\ \text{s.t.} \quad C_0 + K_1 = f(K_0), \quad K_0 \text{ given} \end{cases} \tag{1.52}$$

Noticing that $\begin{bmatrix} \max\limits_{\{C_t,K_{t+1}\}_{t=1}^{\infty}} & \sum_{t=1}^{\infty} \beta^t u(C_t) \\ \text{s.t.} & C_t + K_{t+1} = f(K_t), \quad K_1 \text{ given} \end{bmatrix} = \beta V(K_1)$, we can write (1.52) as

$$V(K_0) = \begin{cases} \max\limits_{\{C_0,K_1\}} \Big\{ u(C_0) + \beta V(K_1) \Big\} \\ \text{s.t.} \quad C_0 + K_1 = f(K_0), \quad K_0 \text{ given} \end{cases} \tag{1.53}$$

or

$$V(K_0) = \max_{\{K_1\}} \Big\{ u(f(K_0) - K_1) + \beta V(K_1) \Big\} \tag{1.54}$$

which is `Bellman's equation`. BE is a functional equation, meaning the unknown is a function $V(\cdot)$.

Jargons: In dynamic programming, as in optimal control:

1. `State variable`: Ex. K_0
2. `Control variable`: Ex. K_1

FONC w.r.t K_1:

$$-u'(f(K_0) - K_1) + \beta V'(K_1) = 0 \tag{1.55}$$

$$u'(f(K_0) - K_1) = \beta V'(K_1) \tag{1.56}$$

Theorem 1.3 (Envelope Theorem) *Let $f(x, a)$ be the a C^1 function of $x \in \mathbb{R}^n$, where a is some exogenously determined parameter, $a \in \mathbb{R}$, and consider the problem of maximizing the function $f(x, a)$. Suppose that $x^*(a)$ is an interior solution, where $x^*(a)$ is a C^1 function of a. Then*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a} f(x^*(a), a) &= \sum_i \frac{\partial f}{\partial x_i}(x^*(a), a) \, \frac{\mathrm{d}x_i}{\mathrm{d}a}(a) + \frac{\partial f}{\partial a}(x^*(a), a) \\ &= \frac{\partial f}{\partial a}(x^*(a), a) \end{aligned} \tag{1.57}$$

References